# PLÜCKER AND STUDY COORDINATES FOR COMPUTATIONAL GEOMETRY

by

Edward Allan Kmett

Thesis

Submitted to the Department of Mathematics

Eastern Michigan University

in partial fulfillment of the requirements

for the degree of

MASTER OF ARTS

in

Mathematics

Thesis Committee:

Ovidiu Calin, PhD, Chair

Walter Parry, PhD

Jayakumar Ramanathan, PhD

Ypsilanti, Michigan

July 1, 2005

# Dedication

To my friends,

the #Coders of IRCnet.

You cemented my obsession with graphics,

visibility and obscure algorithms –

even if I did tend

to bore you guys

with the details.

# Acknowledgements

I would like to thank the members of my thesis committee, in particular my thesis committee chair, Dr. Ovidiu Calin, for his unfailing accessibility and enthusiasm. Dr. Calin, working with you is always a pleasure.

Dr. Walter Parry, you have my gratitude for your willingness to lend a critical ear to whatever problem I might be tackling at the time, your continual insistence on clarity, and for all of your assistance as my graduate advisor. You have the most incredible work ethic I have ever seen.

Dr. Jayakumar Ramanathan, your constant stream of well chosen references on just about every imaginable topic has kept me always reading and always learning since my arrival at Eastern Michigan University. You helped turn me from the path of corporate drudgery and while I may blame you if my addiction to academia leaves me beggared and broken, somehow I don't think I will.

Thank you to my fiancée, Amy Renaud, for putting up with my schedule over the last year and for your unflagging patience and understanding. I would not have been able to accomplish even half of what I have done were it not for your conscientious efforts. I promise to take a vacation with you and relax - some day.

Finally, I would like to thank everyone at the Eastern Michigan University Departments of Mathematics and Computer Science. Your willingness to allow me to break the rules and bypass red tape have allowed me to push my limits and find untapped reserves.

# Abstract

We present a coherent framework for manipulating Plücker and Study coordinates in both $\mathbf{PG}(3, F)$ and $\mathbf{OPG}(3, F)$ with an eye towards their application to stabbing line and transversal problems in computational geometry. We re-derive Study coordinates as a means to diagonalize the quadratic form underlying the Plücker quadric and extend the traditional definition of the Study representative pair to use oriented points facilitating their use in modern oriented elliptic geometry and enabling them to distinguish between a line and its absolute polar. We establish a number of these results using the properties of arbitrary isotropic quadratic spaces enabling us to generalize earlier results for line-line and line-polygon stabbing line problems in $\mathbf{PG}(3, \mathbb{R})$.

# Contents

# Chapter 1: Introduction

*Everything old is new again.*

*–Peter Allen*

Computational geometry is a fairly young interdisciplinary science concerned with the design and analysis of geometric algorithms and data structures. In many ways it brings together some of the oldest areas of mathematics with some of the newest techniques in computer science.

Geometry itself has managed to all but fall out of the college curriculum over the last 60 years, due to pressures from other areas of mathematics and other disciplines, so there is a fairly high barrier to entry to the field of computational geometry. Much of the literature available from when people were first concerned with some of these problems is written in a style nearly inaccessible to a reader with a modern mathematical background.

One of the tools in common usage in computational geometry is the representation of lines in three-dimensional space by their Plücker coordinates. Historically Plücker coordinates are an oddity. They do not generalize well to higher dimensional spaces, but they have a number of nice computational properties that make them especially well suited to solving a wide array of computational problems in three dimensions involving visibility and light transfer.

There is another line representation of historical and (as we will show) computational interest, the Study representative pair or Study coordinates for the line, which offer a number of computational benefits for the same class of problems, but which are generally not known. Eduard Study's original text on the subject, *Geometrie der Dynamen*, published in 1903, was not well received at the time. Using oriented projective geometry allows us to extend the definition of Study coordinates and the Study representative pair to the case of oriented lines, and we explore the surprising connection of Study coordinates to the Plücker quadric.

We attempt here to present a rigorous and coherent framework for calculating with both of these systems of coordinates in a manner accessible to a person without an extensive background in geometry. In doing so, it becomes necessary to introduce a fairly recent development, the use of "oriented" projective and elliptic geometry. Oriented geometry is somewhat less elegant than traditional elliptic geometry. In fact, one of Felix Klein's achievements that is most often cited is his development of elliptic geometry, in which the two points on the opposite sides of a sphere were explicitly identified, precisely to remove this nagging question of orientation. Yet without orientation, many of these problems become intractable in a computational setting.

A reader with a solid background in geometry can probably skip over the second and fourth chapters and refer to them as necessary via the index. The remaining chapters are incremental in complexity, and each introduces notation and conventions built upon in later chapters.

After the geometric preliminaries are dispensed with, we construct a consistent framework for working with the two related systems of coordinates for oriented lines in computational geometry. Most proofs only rely on a solid grasp of linear algebra. However, some of the later ones require a bit of background in analysis and a familiarity with quadratic forms. Finally, there are appendices of historical concerns, notational considerations, and of other results involving Plücker or Study coordinates that are either of only cursory interest or would call for the incorporation of too much additional material to be effectively presented.

# Chapter 2: Tools from Classical Geometry

*I must create a system myself or be enslaved by another man's.*

*–William Blake*

While computational geometry uses projective and elliptic geometry heavily, these topics have managed to become all but non-existent in the curriculum. We introduce the concepts of a projective space axiomatically both in order to provide the reader with a basic grasp of the geometric tools and to establish a consistent vocabulary for use in later chapters. The choice of terminology and notation follows that of Beutelspacher and Rosenbaum.[1] This treatment is by no means exhaustive and is intended to prepare for working with structures built over a small number of geometries exclusively. Many classical theorems in the field are left completely unmentioned, and little attempt is made at full generality.

## 2.1 Defining a projective space

**Definition 2.1.1** *A **geometry** is an ordered pair,* $\mathbf{G} = (\Omega, I)$ *, where* $\Omega$ *is a set and* $I$ *is a relation on* $\Omega$ *that is symmetric and reflexive. We say that* $I$ *is the **incidence relation** of the geometry* $\mathbf{G}$*. Given* $x, y \in \Omega$*, if* $(x, y) \in I$ *then we say that* $x$ *and* $y$ *are **incident**.*

**Definition 2.1.2** *Let* $\mathbf{G} = (\Omega, I)$ *be a geometry. A **flag** of* $\mathbf{G}$ *is a subset of* $\Omega$ *such that every element in the subset is incident to every other element in the subset. A flag* $F$ *is called*

***maximal*** *if there does not exist* $x \in \Omega$ *such that* $F \cup \{x\}$ *is also a flag. We refer to elements of* $\Omega$ *as* ***geometric primitives*** *or* ***flats****.*

**Definition 2.1.3** *We say a geometry* $\mathbf{G} = (\Omega, I)$ *has* ***rank*** $n$ *if* $\Omega$ *can be partitioned into an ordered set* $(\Omega_1, \ldots, \Omega_n)$ *such that every maximal flag of* $\mathbf{G}$ *contains exactly one element from* $\Omega_i$*, for* $i \in \{1, \ldots, n\}$*.*

**Definition 2.1.4** *If* $\mathbf{G} = (\Omega, I)$ *is a geometry of rank* $2$ *where* $\Omega$ *can be partitioned into* $(\mathcal{P}, \mathcal{B})$ *in the manner just described, then we say that* $\mathbf{G}$ *is an* ***incidence structure*** *and we write* $\mathbf{G} = (\mathcal{P}, \mathcal{B}, I)$*. We call the elements of* $\mathcal{P}$ ***points****, and the elements of* $\mathcal{B}$ ***blocks****.*

Usually we will be concerned with the case where the incidence structure describes a relationship between points and **lines**, so we may refer to blocks and lines interchangeably.

**Definition 2.1.5** *Given a proposition* $A$ *concerning an incidence structure, we obtain the proposition* $A^{\Delta}$ ***dual*** *to* $A$ *by interchanging the words "point" and "line."*

**Definition 2.1.6** *Given an incidence structure* $\mathbf{G} = (\Omega_1, \Omega_2, I)$*, we define the incidence structure* $\mathbf{G}^{\mathbf{\Delta}}$ ***dual*** *to* $\mathbf{G}$ *by* $\mathbf{G}^{\mathbf{\Delta}} := (\Omega_2, \Omega_1, I)$*.*

By this definition, clearly $(\mathbf{G}^{\mathbf{\Delta}})^{\mathbf{\Delta}} = \mathbf{G}$.

**Definition 2.1.7** *Let* $\mathbf{G} = (\mathcal{P}, \mathcal{L}, I)$ *be an incidence structure. Let* $g, h \in \mathcal{L}$ *be two distinct lines. We say the lines* $g$ *and* $h$ ***intersect*** *if there exists a point* $P \in \mathcal{P}$ *such that* $(g, P) \in I$ *and* $(h, P) \in I$*.*

**Definition 2.1.8** *A* ***projective space*** $\mathbf{P} = (\Omega, I)$ *is an geometry of rank* $n \geq 2$*, where the contents of two of the partitions of* $\Omega$ *are singled out as* ***points*** *and* ***lines*** *respectively in which the following axioms hold:*

***Axiom 1*** *For any two distinct points P and Q there exists exactly one line that is incident with both P and Q. We denote this line PQ. This axiom is known as the* ***line axiom****.*

***Axiom 2*** *Let A, B, C and D be four distinct points such that AB intersects CD. Then AC also intersects BD. This axiom is usually called the* ***Veblen-Young axiom****.*

***Axiom 3*** *Any line is incident with at least three points.*

***Axiom 4*** *There are at least two lines.*

Traditionally one distinguishes between a degenerate projective space, which only satisfies the first three axioms given above, and a non-degenerate projective space, which satisfies all four. Since we are only concerned with non-degenerate projective spaces, we simply refer to them as projective spaces and assume all four axioms hold.

### 2.2 Additional properties of a projective plane

**Definition 2.2.1** *A* ***projective plane*** *is a projective space in which any two lines have at least one point in common. This is a stronger statement than the Veblen-Young axiom and so replaces it.*

This counter-intuitive definition is where the traditional Euclidean geometry and the geometry of a projective plane start to diverge, but it gives rise to the most startling property of projective geometry, the **Principle of Duality**, defined in Theorem 2.2.7 below. We build up to this by noting that the duals of each of the axioms for a projective space hold.

**Lemma 2.2.2** ***(Axiom** $1^{\Delta}$***)*** *For any two distinct lines in a projective plane there is exactly one point that is incident with both.*

*Proof:* Let $g$ and $h$ be distinct lines. By the definition of a projective plane, $g$ and $h$ are incident with at least one common point. Assume there exists a pair of distinct points $P$ and $Q$, each incident with both $g$ and $h$. By the line axiom, there exists only one line through $P$ and $Q$, so $g = h$. Contradiction. Therefore $g$ and $h$ intersect at exactly one point. ■

**Lemma 2.2.3** ***(Axiom** $2^{\Delta}$***)*** *Any two points are incident to at least one line.*

*Proof:* This is an immediate consequence of the first axiom. ■

**Lemma 2.2.4** *Given a point $P$ in a projective space, there exists a line not incident with $P$.*

*Proof:* Let $P$ be a point. Assume that all lines are incident with $P$. Then there exist distinct lines $g_1$ and $g_2$ through $P$. By the third axiom there exist points $G_1$ and $G_2$ on $g_1$ and $g_2$ respectively, where $G_i \neq P$. If $G_1G_2$ is a line through $P$ then we violate the first axiom or that $g_1$ and $g_2$ are distinct, so $G_1G_2$ is a line not incident with $P$. Contradiction. So given $P$ there exists a line not incident with $P$. ■

**Lemma 2.2.5** ***(Axiom** $3^{\Delta}$***)*** *Any point is incident with at least three lines.*

*Proof:* Let $P$ be a point. Then by Lemma 2.2.4 there exists a line $g$ which is not incident with $P$. By the third axiom there exists at least three distinct points $A$,$B$, and $C$ incident with $g$. So $AP$, $BP$, and $CP$ are three distinct lines through $P$. ■

**Lemma 2.2.6** ***(Axiom** $4^{\Delta}$***)*** *There exist two distinct points.*

*Proof:* There are at least two lines by fourth axiom, and every line is incident with at least three points, so there are easily two distinct points. ■

Since Axioms $1^\Delta$, $2^\Delta$, $3^\Delta$, and $4^\Delta$ hold for every projective plane, it follows that if $P$ is a projective plane, then so is $P^\Delta$.

**Theorem 2.2.7** *(**Principle of Duality**) If a proposition A holds for all projective planes then the dual proposition $A^\Delta$ also holds for all projective planes.*

*Proof:* Let $P$ be a projective plane. If proposition $A$ holds in all projective planes then in particular proposition $A$ holds for $P^\Delta$. It follows that proposition $A^\Delta$ holds for $(P^\Delta)^\Delta = P$.

■

### 2.3 Constructing a projective space over a vector space

We are concerned with a specific type of projective space that recommends itself to efficient processing by means of linear algebra. To that end, we describe the construction of a projective space over a vector space as follows.

**Definition 2.3.1** *If V is a vector space of dimension $(d + 1)$, where $d + 1 \geq 3$, over a field F, then we define $\mathbf{P}(V) = (\Omega, I)$ where:*

- *The elements of $\Omega$ are the proper nontrivial linear subspaces of $V$.*
- *$(\alpha, \beta) \in I$ if and only if $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$.*

**Theorem 2.3.2** *If V is a vector space of dimension $d + 1$, where $d + 1 \geq 3$ then $\mathbf{P}(V) = (\Omega, I)$ is a geometry of rank d with the partition $(\Omega_1, ..., \Omega_d)$, where*

$$\Omega_i = \{U : U \textit{ is a subspace of } V \textit{ and } dim(U) = i\} \textit{ for } i \in \{1, \ldots, d\}.$$

*Proof:* It is clear that $\mathbf{P}(V)$ is a geometry. We must show that it has rank $d$.
Assume there exists a flag in $\mathbf{P}(V)$, which contains elements $\alpha, \beta \in \Omega_i$ for some $i \in \{1, \ldots, d\}$. Because $\alpha$ and $\beta$ have the same dimension and one contains the other, they are the same subspace. So there is at most one element from $\Omega_i$ in any given flag of $\mathbf{P}(V)$.
On the other hand, assume there exists a maximal flag $\mathcal{F}$ in $\mathbf{P}(V)$, which does not contain an element of each partition set of $\Omega$. Choose $i$ such that $\Omega_i \cap \mathcal{F} = \emptyset$.
If there exists a $j < i$ such that $\mathcal{F} \cap \Omega_j \neq \emptyset$, then let $j_0$ be the largest such $j$ and let $\beta$ be a basis for a subspace in $\mathcal{F} \cap \Omega_{j_0}$. Otherwise let $\beta = \emptyset$. If there exists a $k > i$ such that $\mathcal{F} \cap \Omega_k \neq \emptyset$, then let $k_0$ be the smallest such $k$ and let $U \in \mathcal{F} \cap \Omega_{k_0}$. Otherwise let $U = V$.
Since $\beta$ is a linearly independent subset of $U$, $\beta$ can be extended to a basis $\beta'$ of $U$. Let $W$ be the span of any $i$-element subset of $\beta'$ which contains $\beta$.
Clearly span $\beta \subset W \subset U$. So any subset of span $\beta$ is a subset of $W$, and $W$ is a subset of any superset of $U$, so $\mathcal{F} \cup \{W\}$ is a flag. Therefore $\mathcal{F}$ is not a maximal flag. So any maximal flag of $\mathbf{P}(V)$ contains exactly one element of $\Omega_i$ for $i \in \{1, \ldots, d\}$. ■

**Theorem 2.3.3** *If $V$ is a vector space of dimension at least 3 then $\mathbf{P}(V) = (\Omega, I)$ with the ordered partition $(\Omega_1, ..., \Omega_n)$ is a projective space with point set $\Omega_1$ and line set $\Omega_2$.*

*Proof:* We simply need to verify that each of the axioms required of a projective space holds.
*For any two distinct points $P$ and $Q$ there exists exactly one line that is incident with both $P$ and $Q$.* Because $P$ and $Q$ are distinct one-dimensional linear subspaces, there exists a unique two-dimensional linear subspace that contains them both.
*Let $A$, $B$, $C$ and $D$ be four points such that $AB$ intersects $CD$. Then $AC$ also intersects $BD$.* If $AB$ intersects $CD$, then either $AB = CD$ or they intersect at a single point $P$. If $AB = CD$ then $AB = CD = AC = BD$ and so $AC$ intersects $BD$. Otherwise $AB$ and $CD$

intersect at $P$. Let $\mathbf{p}$ be a vector such that $\operatorname{span}\{\mathbf{p}\} = P$. Then we can extend $\{\mathbf{p}\}$ to a basis $\beta$ for $AB$. Similarly we construct a basis $\gamma$ for $CD$ that contains $\mathbf{p}$. Because $\{\mathbf{p}\} = \beta \cap \gamma$, we know $\beta \cup \gamma$ is a set of three vectors, which span a three-dimensional linear subspace $\pi$ that contains the points $A$, $B$, $C$ and $D$. So, $AC$ and $BD$ are contained in $\pi$. Since $\pi$ is a three-dimensional linear space and $AC$ and $BD$ are distinct two-dimensional linear subspaces of $\pi$, they must have a one-dimensional linear subspace in common. Therefore $AC$ intersects $BD$.

*Any line is incident with at least three points.* Given a line $\ell$, there exists a basis $\beta = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1\}$ such that $\operatorname{span}\beta = \ell$. Then $\operatorname{span}\{\boldsymbol{\beta}_0\}$, $\operatorname{span}\{\boldsymbol{\beta}_1\}$, and $\operatorname{span}\{\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1\}$ are three points on the line.

*There are at least two lines.* This is the reason for the restriction that the vector space must have dimension $\geq 3$. Let $\beta = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2\}$ be an arbitrary basis for any three-dimensional linear subspace of $V$. Then $\operatorname{span}\{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1\}$ and $\operatorname{span}\{\boldsymbol{\beta}_0, \boldsymbol{\beta}_2\}$ are distinct lines. ■

**Definition 2.3.4** *If a projective space* $\mathbf{P}$ *is also a geometry of rank* $d$*, we say that it is a projective space of* ***dimension*** $d$*.*

**Definition 2.3.5** *We say that the projective space* $\mathbf{P}(V)$ *is* ***constructed over*** $V$ *and* ***coordinatized*** *by* $F$*.*

**Definition 2.3.6** *We define a* $k$*-****flat*** *in* $\mathbf{P}(V) = (\Omega, I)$ *as an element of the partition set* $\Omega_{k+1}$*, or in other words any* $(k+1)$*-dimensional linear subspace of* $V$*. We refer to* 0-, 1- *and* 2-*flats as* ***points***, ***lines***, *and* ***planes*** *respectively. In a vector space of dimension* $n$*, we refer to an* $(n-1)$*-dimensional linear subspace as a* ***hyperplane***.

We borrow the following terminology from Stolfi.[16]

**Definition 2.3.7** *In the context of* $\mathbf{P}(V)$ *we define the* ***meet*** *of two flats* $\alpha$ *and* $\beta$ *to be the linear subspace of their intersection and write it as* $\alpha \wedge \beta$.

**Definition 2.3.8** *In the context of* $\mathbf{P}(V)$ *we define the* ***join*** *of two flats* $\alpha$ *and* $\beta$ *to be the linear subspace of* $V$ *generated by the elements of* $\alpha \cup \beta$. *We write this as* $\alpha \vee \beta$.

**Definition 2.3.9** *We define the* ***vacuum*** $\mathbf{F}$ *to be the trivial zero-dimensional subspace* $\{0\}$ *that may arise from taking the meet of flats in a projective space.*

Note that for any flat $P$,

$$\begin{aligned} \mathbf{F} \wedge P = P \wedge \mathbf{F} = \mathbf{F}, \\ \mathbf{F} \vee P = P \vee \mathbf{F} = P. \end{aligned} \tag{2.3.1}$$

This lends (hopefully) a bit of humor to this choice of notation and provides a convenient mnemonic device from elementary truth tables.

**Definition 2.3.10** *We define the* ***universe*** *to be all of our underlying vector space* $V$.

**Definition 2.3.11** *Given a proposition* $A$ *concerning a geometry* $\mathbf{G}$ *of rank* $n$ *we obtain the proposition* $A^{\Delta}$ ***dual*** *to* $A$ *by interchanging the words "meet" and "join", replacing each reference to a* $k$*-flat with a reference to an* $(n-k)$*-flat, and swapping the words "vacuum" with "universe."*

**Notation 2.3.12** *We are primarily interested in projective spaces constructed over the vector space* $F^{n+1}$ *coordinatized by* $F$. *We adopt the standard notation of referring to this projective space as* $\mathbf{PG}(n, F)$.

### 2.4 Reguli and transversals

**Definition 2.4.1** *If a line $\ell$ exists that stabs through every line in a set $\mathcal{L}$ of lines at exactly one point each, that line is called a* ***transversal*** *of the set $\mathcal{L}$.*

**Definition 2.4.2** *A set of flats in a projective space are* ***skew*** *if no two flats in the set contain a common point.*

It follows no two lines in a set of skew lines lie in the same plane.

**Definition 2.4.3** *A* ***regulus*** *is a set of three skew lines.*

**Lemma 2.4.4** *Given two skew lines $\ell_1$, $\ell_2$ in* $\mathbf{PG}(3, \mathbb{R})$ *and a point not contained in either of the two lines there exists a unique line $\ell$ that intersects $\ell_1$, $\ell_2$, and $P$.*

*Proof:* Consider the plane $\Pi = \ell_1 \vee P$. Since $\Pi$ contains $\ell_1$ and we know that $\ell_1$ and $\ell_2$ are skew, $\ell_2$ is not contained in the plane. However, since we are in a four dimensional vector space, and we know $\ell_2$ is a two-dimensional subspace and $\Pi$ is a three-dimensional subspace, $Q = \ell_2 \wedge \Pi$ must be a point. Then the line $m = P \vee Q$ is a line that intersects $\ell_2$ which is contained in $\Pi$. Since $\ell_1$ and $m$ are contained in the same projective plane, and $\ell_1 \neq m$, we know they intersect at a single point. So there exists a unique transversal $m$ of $\ell_1$ and $\ell_2$ that passes through any arbitrary point $P$ in space. ■

**Corollary 2.4.5** *Given three skew lines $\ell_1$, $\ell_2$, $\ell_3$ and a point on one of the three, there exists a unique transversal through all three that passes through that point.*

# Chapter 3: Oriented Projective Geometry

## 3.1 Orienting a projective space

The lack of directionality in a projective space is somewhat frustrating for the purposes of computational geometry and computer graphics. To give lines and planes a meaningful orientation, we introduce the concept of an oriented projective space.

**Definition 3.1.1** *An* ***oriented projective space*** $\mathbf{O} = (\mathbf{P}, \Omega_O, |\cdot|)$ *consists of an underlying projective space* $\mathbf{P} = (\Omega_P, I)$*, a set* $\Omega_O$ *of* ***oriented flats****, and a function* $|\cdot|$ *that maps elements of* $\Omega_O$ *into elements of* $\Omega_P$*, such that exactly two oriented flats* $+\alpha$ *and* $-\alpha$ *in* $\Omega_O$ *map to the same flat* $\alpha$ *in* $\Omega_P$*.*

**Definition 3.1.2** *Given an oriented flat* $\alpha$ *in an oriented projective space* $\mathbf{O}$*, we define* $-\alpha$ *to be the other oriented flat with* $|\alpha| = |-\alpha|$*.*

Note that because the vacuum and universe are not flats, there is no oriented version of them contained in $\Omega_O$.

Because an oriented projective space is constructed from a projective space, we can use all of the structure associated with the underlying projective space. Similarly, we refer to incidence for oriented flats in terms of the underlying oriented flats they map to via $|\cdot|$.

**Example 3.1.1** *We say two oriented flats* $\alpha$ *and* $\beta$ *are incident if the flats* $|\alpha|$ *and* $|\beta|$ *are incident.*

As we are concerned only with oriented projective spaces constructed over vector spaces, we define matters directly in those terms. From here on, we will assume that the projective space in the definition above is constructed over a vector space $V$.

**Notation 3.1.3** *If* $\mathbf{v}$ *is a nonzero vector in* $V$ *then we let* $P_{\mathbf{v}} = \operatorname{span}\{\mathbf{v}\}$*, a point in our projective space. Without previous mention of the vector* $\mathbf{v}$*, if we refer to a point* $P_{\mathbf{v}}$ *in our projective space, then we mean that it arises in this way from a vector* $\mathbf{v}$*.*

Furthermore, we shall assume that the field $F$ is an ordered field when dealing with oriented projective spaces, allowing us to refer to positive and negative elements of $F$ unambiguously.

An alternative full formal definition of an oriented projective space and an oriented generalization of the meet and join operations appears in several papers by Stolfi.[16,17]

**Definition 3.1.4** *In an oriented projective space each* ***oriented flat*** $\alpha$ *is an equivalence class of ordered bases of the underlying flat* $|\alpha|$*. Ordered bases of* $|\alpha|$ *are equivalent if their change of basis matrices have positive determinant.*

Clearly, there are only two oriented flats associated with $|\alpha|$, so these satisfy the requirements for oriented flats given above.

As a consequence of the above definition, in an oriented projective space each **oriented point** is an equivalence class of vectors defined so that two vectors are in the same class if one is a positive scalar multiple of the other. The special equivalence class $\mathbf{F}$ containing only $\{\mathbf{0}\}$ is not an oriented point, as it is not a basis for a one-dimensional linear subspace. As before, we refer to this degenerate case as the vacuum.

We use the notation $P_{\mathbf{v}}$ for oriented points as well as for unoriented points. Context will make the meaning clear.

### 3.2 Homogeneous coordinates

Because many of the definitions in the following sections are virtually identical for the oriented and conventional cases, when a proof applies to both with only minor changes necessary to make the definitions apply in the oriented case, we parenthesize these changes. Note that because an oriented projective space is constructed from a projective space, the weaker definition can still be used when orientation is not at issue.

**Definition 3.2.1** *Let* **G** *be an (oriented) projective space constructed over a vector space $V$ and let $\beta$ be a basis for $V$. For $\mathbf{v} \in V$, we say that the components of column vector $[\mathbf{v}]_\beta$ are* ***homogeneous coordinates*** *for the (oriented) point $P_\mathbf{v}$ with respect to $\beta$. If $\beta$ is unspecified then we fix a basis for $V$, preferably the canonical basis if one exists, and use it consistently.*

**Notation 3.2.2** *We refer to the oriented projective space constructed over* $\mathbf{P}(V)$ *as* $\mathbf{OP}(V)$*. Similarly we refer to the oriented projective space constructed over* $\mathbf{PG}(n, F)$ *as* $\mathbf{OPG}(n, F)$*.*

The projective space $\mathbf{PG}(3, \mathbb{R})$ is familiar turf to programmers in the field of computer graphics; it is the space in which traditional computer graphics operations are usually executed. The $4 \times 4$ matrices in any introductory computer graphics text that represent rotation, translations, and perspective projection all give mappings that preserve the structure of $\mathbf{PG}(3, \mathbb{R})$ as a projective space.

### 3.3 Polarities

While it is possible to construct a purely projective geometry that contains oriented flats, and this has been done quite successfully in a computational setting, this is not necessary for our purposes and interferes with the operations we need to define, so we turn to elliptic geometry for another tool.

**Definition 3.3.1** *An (oriented)* ***correlation*** *on an (oriented) projective space is bijective map on (oriented) flats that maps (oriented) k-flats into (oriented)* $(n-k)$*-flats in the same projective space, while preserving incidence (and orientation).*

**Example 3.3.2** *If* $*$ *refers to the application of the correlation to a flat, then this definition states that* $A$ *and* $B$ *are incident if and only if* $A^*$ *and* $B^*$ *are incident.*

**Definition 3.3.2** *We define a* ***polarity*** *as a correlation which is its own inverse.*

**Definition 3.3.3** *We define an (oriented)* ***elliptic space*** *to be an (oriented) projective space with a fixed unoriented polarity, called the* ***absolute polarity****.*

There can exist more than one polarity on a given projective space. However, for our purposes we are only concerned with the absolute polarity. Any reference to polarity refers to the absolute polarity and is not necessarily extensible to an arbitrarily constructed polarity.

When we are working with a projective geometry constructed over an inner product space, there is a particular polarity that is readily apparent to us:

**Definition 3.3.4** *Let* $\alpha$ *be a flat in* $\mathbf{P}(V)$*, where* $V$ *is a nondegenerate inner product space. We define the flat* $\alpha^{\Delta}$ ***polar*** *to* $\alpha$ *to be the linear subspace of vectors orthogonal to* $\alpha$ *under the inner product of* $V$*.*

Many of the ways in which we use this polarity rely solely upon its existence and this cannot be guaranteed in $\mathbf{P}(V)$ alone without the stipulation that $V$ be an inner product space. Because $\mathbf{PG}(n, F)$ are constructed over $F^{n+1}$, which allows the use of the standard inner product, we can safely use this construction there. Rather than clutter the text with another notation or repeated caveats that $V$ be an inner product space, or that we have a given polarity, we simply limit our attention to $\mathbf{PG}(n, F)$ and assume the polarity just given is the absolute polarity for the space.

**Notation 3.3.5** *We adopt the convention that the tuple* $[a_0, \ldots, a_n]$ *refers to a row vector, while* $(a_0, \ldots, a_n)$ *refers to the column vector* $[a_0, \ldots, a_n]^T$.

There are two oriented correlations that arise naturally when considering an oriented projective space and that are consistent with the absolute polarity for the underlying projective space.

**Definition 3.3.6** *Let* $\alpha$ *be an oriented flat that contains the ordered basis* $(\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$. *We define the* ***right complement*** *of the oriented flat* $\alpha$ *to be the oriented flat* $\alpha^{\vdash}$, *the equivalence class of bases for* $|\alpha|^{\Delta}$ *that contains the basis* $(\boldsymbol{\alpha}_1^{\vdash}, ..., \boldsymbol{\alpha}_m^{\vdash})$ *such that a fixed basis* $\beta$ *for the underlying vector space can be obtained from* $(\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n, \boldsymbol{\alpha}_1^{\vdash}, ..., \boldsymbol{\alpha}_m^{\vdash})$ *by a matrix of positive determinant.*

**Definition 3.3.7** *Let* $\alpha$ *be an oriented flat that contains the ordered basis* $(\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$. *We define the* ***left complement*** *of the oriented flat* $\alpha$ *to be the oriented flat* $\alpha^{\dashv}$, *the equivalence class of bases for* $|\alpha|^{\Delta}$ *that contains a basis* $(\boldsymbol{\alpha}_1^{\dashv}, ..., \boldsymbol{\alpha}_m^{\dashv})$ *such that the fixed basis* $\beta$ *for the underlying vector space can be obtained from* $(\boldsymbol{\alpha}_1^{\dashv}, ..., \boldsymbol{\alpha}_m^{\dashv}, \boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$ *by a matrix of positive determinant.*

Clearly, the act of taking the left or right complement of an oriented flat is the application of a correlation. Furthermore, both of these correlations are consistent with the absolute polarity for the elliptic space. So $|\alpha^{\vdash}| = |\alpha^{\dashv}| = |\alpha|^{\Delta}$. This allows us to use these operations in an unoriented projective space without further comment.

If $\dim(|\alpha|) \cdot \dim(|\alpha|^{\Delta})$ is even, then $\alpha^{\vdash} = \alpha^{\dashv}$. Otherwise $\alpha^{\vdash} = -\alpha^{\dashv}$. For instance, in the case of an oriented line $\ell$ in $\mathbf{OPG}(3, F)$, this product is 4, so $\ell^{\vdash} = \ell^{\dashv}$. However, for an oriented point or oriented plane $P$ in $\mathbf{OPG}(3, F)$ this product is 3, so $P^{\vdash} = -P^{\dashv}$. This motivates the following definition:

**Definition 3.3.8** *Let $\alpha$ be an oriented flat. If $\dim(|\alpha|) \cdot \dim(|\alpha|^{\Delta})$ is even, then we define the oriented flat $\alpha^{\Delta}$ polar to $\alpha$ to be the unique oriented flat $\alpha^{\Delta} = \alpha^{\dashv} = \alpha^{\vdash}$.*

### 3.4 Tangential coordinates

**Definition 3.4.1** *Let* **G** *be a d-dimensional (oriented) elliptic space constructed over a standard inner product space $V$. We may specify an (oriented) hyperplane $\Pi$ using the homogeneous coordinates of an (oriented) point $\Pi^{\dashv}$ that is the left orthogonal complement to $\Pi$. Borrowing terminology from Coxeter[3] we refer to the row vector formed from the homogeneous coordinates for $\Pi^{\dashv}$ as **tangential coordinates** for $\Pi$.*

This choice of terminology is somewhat unfortunate in that these coordinates might more accurately be dubbed “orthogonal” or “normal” coordinates, but the convention of referring to them as tangential is fairly well established.

**Definition 3.4.2** *We refer to an oriented point whose first nonzero homogeneous coordinate is positive as a **positively oriented point**. Similarly we define **negatively oriented** points as those whose first nonzero homogeneous coordinate is negative.*

**Definition 3.4.3** *By the same convention, we refer to an oriented hyperplane whose first nonzero tangential coordinate is positive as a **positively oriented hyperplane**. Similarly we define negatively oriented hyperplanes and the orientation of a hyperplane as before.*

**Notation 3.4.4** *We adopt the convention that* $\Pi_{\boldsymbol{\pi}}$ *refers to the (oriented) hyperplane, which is the (right orthogonal) complement to the (oriented) point* $P_{\boldsymbol{\pi}}$.

**Definition 3.4.5** *Given an oriented hyperplane* $\Pi_{\boldsymbol{\pi}}$ *in an oriented projective space, we define* $H^+_{\Pi_\pi} = \{P_{\mathbf{v}} : \langle \mathbf{v}, \boldsymbol{\pi} \rangle \geq 0\}$ *and* $H^-_{\Pi_\pi} = \{P_{\mathbf{v}} : \langle \mathbf{v}, \boldsymbol{\pi} \rangle \leq 0\}$, *to be the set of oriented points in the positive and negative half-spaces associated with* $\Pi_\pi$.

**Claim 3.4.6** *Clearly* $H^+_\Pi = H^-_{-\Pi}$, *demonstrating the distinction between positively and negatively oriented planes. It is also easy to see that if* $P$ *is an oriented point not incident with* $\Pi$ *then* $P \in H^+_\Pi$ *if and only if* $-P \in H^-_\Pi$.

## 3.5 Oriented meet and join operations

We can only partially extend the meet and join operations to oriented flats in a manner that is consistent with the meet and join operations on the underlying flats.
See Stolfi[16] for an alternate definition, which includes additional mechanisms for handling flats of indeterminant orientation that is suited to implementation in software, where special cases and undefined quantities are awkward to work around.

**Definition 3.5.1** *Given two oriented flats* $\alpha$ *and* $\beta$ *that contain the ordered bases* $(\boldsymbol{\alpha}_1, ..., \boldsymbol{\alpha}_n)$ *and* $(\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_m)$ *such that* $|\alpha| \cap |\beta| = \{0\}$, *we define their* ***join***, *the oriented flat* $\alpha \vee \beta$, *to be the oriented flat which contains the ordered basis* $(\boldsymbol{\alpha}_1, ..., \alpha_n, \boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_m)$.

**Definition 3.5.2** *Given two oriented flats* $\alpha$ *and* $\beta$ *in an oriented elliptic space such that* $|\alpha|^\Delta \cap |\beta|^\Delta = \{0\}$, *we define their* ***meet***, *the oriented flat* $\alpha \wedge \beta$, *in terms of the oriented join and right orthogonal complement as*

$$\alpha \wedge \beta \coloneqq (\alpha^\vdash \vee \beta^\vdash)^\vdash. \tag{3.5.1}$$

It can be readily verified that the following relationships hold for the oriented $a$-flats $\alpha$ and oriented $b$-flat $\beta$ in **OPG**$(n, F)$ by using elementary linear algebra:

- $\alpha \vee (-\beta) = -(\alpha \vee \beta)$,
- $\alpha \vee \beta = (-1)^{ab}(\beta \vee \alpha)$,
- $\alpha \wedge (-\beta) = -(\alpha \wedge \beta)$,
- $\alpha \wedge \beta = (-1)^{ab}(\beta \wedge \alpha)$,
- $\alpha^{\dashv} = (-1)^{a(n-a)}\alpha^{\vdash}$,
- $(\alpha^{\dashv})^{\vdash} = (\alpha^{\vdash})^{\dashv} = \alpha$.

Note that we may consistently use either left or right complement because if $\dim(|\alpha|) = a$ and $\dim(|\beta|) = b$ and our elliptic space is constructed over an $n$ dimensional standard inner product space then

$$\begin{aligned}
(\alpha^{\vdash} \vee \beta^{\vdash})^{\vdash} &= (-1)^{(a+b)(n-a-b)}(\alpha^{\vdash} \vee \beta^{\vdash})^{\dashv} \\
&= (-1)^{(a+b)(n-a-b)}((-1)^{a(n-a)}\alpha^{\dashv} \vee (-1)^{b(n-b)}\beta^{\dashv})^{\dashv} \\
&= (-1)^{an+bn-aa-ab-ba-bb+an-aa+bn-bb}(\alpha^{\dashv} \vee \beta^{\dashv})^{\dashv} \\
&= (-1)^{2an+2bn-2aa-2bb-2ab}(\alpha^{\dashv} \vee \beta^{\dashv})^{\dashv} \\
&= 1^{an+bn-aa-bb-ab}(\alpha^{\dashv} \vee \beta^{\dashv})^{\dashv} \\
&= (\alpha^{\dashv} \vee \beta^{\dashv})^{\dashv}.
\end{aligned} \tag{3.5.2}$$

# Chapter 4: Notions of Infinity

## 4.1 Flats at infinity

Now we return to the distinction that was drawn between Euclidean and projective geometry earlier. We stated that in a projective plane, any two lines intersect at a point. On one level this definition runs counter to geometry as learned in high school, where we can have parallel lines. On another level, it is just another way to refer to the same phenomenon.

**Definition 4.1.1** *Let* **G** *be an (oriented) elliptic space over a standard inner product space* $V$*. If* $\beta = (\beta_0, \ldots, \beta_d)$ *is an ordered orthogonal basis for* $V$ *then we denote by* $H^\infty_\beta$ *the hyperplane* $\Pi_{\beta_0}$*. We call* $H^\infty_\beta$ *the* ***hyperplane at infinity*** *with respect to* $\beta$*. As usual, if* $\beta$ *is unspecified, we refer to the standard basis for V.*

**Example 4.1.3** *In* $\mathbf{PG}(3, F)$, $\mathbf{v} \in H^\infty$ *if and only if* $v_0 = 0$.

**Definition 4.1.2** *If* $U$ *is a k-flat incident with* $H^\infty$*, then we say that* $U$ *is a* ***k-flat at infinity***.

**Example 4.1.4** *If* $U$ *is a two-dimensional linear subspace of* $H^\infty$*, then we say that* $U$ *is a* ***line at infinity***.

### 4.2 Affine coordinates

**Definition 4.2.1** *If U is a k-flat and it is not a k-flat at infinity, then we say that U is an* ***affine k-flat****.*

Note that every affine $k$-flat, where $k > 0$, contains a single $(k-1)$-flat at infinity.

**Definition 4.2.2** *Let* $\mathbf{G}$ *be an elliptic space of dimension n constructed over a standard inner product space V with ordered orthogonal basis* $\beta$*. Let P be an affine point in* $\mathbf{G}$*. Then P has homogeneous coordinates* $(a_0, \ldots, a_n)$ *with respect to* $\beta$ *and because P is not a point at infinity,* $a_0 \neq 0$*. We define the* ***affine coordinates*** *of P with respect to* $\beta$ *to be the vector*

$$\widetilde{\mathbf{a}} = \left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right). \tag{4.2.1}$$

**Example 4.2.5** *Consider* $\mathbf{PG}(2, F)$*. Let P,Q,R and S have affine coordinates* $(0,1)$,$(0,2)$,$(1,0)$*, and* $(1,2)$ *respectively. This tells us that the line* $P \vee Q$ *is spanned by* $\{(1,0,1),(1,0,2)\}$ *and the line* $R \vee S$ *is spanned by* $\{(1,1,0),(1,1,2)\}$*. Therefore the point* $(P \vee Q) \wedge (R \vee S)$ *is spanned by* $\{(0,0,2)\}$ *and since its first homogeneous coordinate is 0, it is a point at infinity. In other words the fact that in the Euclidean plane the lines y=0, and y=1 are parallel is represented by the fact that the intersection between these two lines occurs at a point without affine coordinates.*

# Chapter 5: Quadratic Spaces and Isotropic Vectors

Before we can explore the Plücker quadric, it becomes necessary to review some infrequently used terminology involving quadratic spaces.

## 5.1 Quadratic spaces

**Definition 5.1.1** *A tuple* $(Q, V)$ *consisting of a quadratic form Q and a vector space V is called a **quadratic space**.*

**Definition 5.1.2** *Given a quadratic space* $(Q, V)$*, a vector* $\mathbf{v} \in V$ *is defined to be **isotropic** if* $Q(\mathbf{v}) = 0$*, and* $\mathbf{v} \neq 0$.

**Definition 5.1.3** *If a quadratic space* $(Q, V)$ *contains an isotropic vector then we say that* $(Q, V)$ *is an **isotropic quadratic space**. Otherwise we say that the quadratic space is **anisotropic**.*

## 5.2 Signatures

**Definition 5.2.1** *Let* $(Q, V)$ *be a quadratic space where B is the bilinear form underlying Q. If there exists an ordered basis* $\beta = (\boldsymbol{\beta}_0, ..., \boldsymbol{\beta}_n)$ *for V which is orthogonal under B such that*

$$Q(\boldsymbol{\beta}_0) = \cdots = Q(\boldsymbol{\beta}_{i-1}) = 1, \tag{5.2.1}$$

$$Q(\boldsymbol{\beta}_i) = \cdots = Q(\boldsymbol{\beta}_{i+j-1}) = -1, \tag{5.2.2}$$

$$Q(\boldsymbol{\beta}_{i+j}) = \cdots = Q(\boldsymbol{\beta}_n) = 0 \tag{5.2.3}$$

*then we say that* $(Q, V)$ *is a* ***quadratic space of signature*** $(i, j)$*, and that* $Q$ *is a* ***quadratic form of signature*** $(i, j)$*.*

**Definition 5.2.2** *A quadratic space* $(Q, V)$ *of signature* $(n, 0)$ *is said to be* ***positive semi-definite*** *if* $dim(V) \geq n$ *and* ***positive definite*** *if* $dim(V) = n$*.*

**Definition 5.2.3** *Similarly, a quadratic space* $(Q, V)$ *of signature* $(0, n)$ *is said to be* ***negative semi-definite*** *if* $dim(V) \geq n$ *and* ***negative definite*** *if* $dim(V) = n$*.*

It is easy to see that if a quadratic space is positive definite or negative definite, than it is anisotropic.

## 5.3 Orthogonal isotropic vectors and quotient spaces

**Definition 5.3.1** *Given a quadratic space* $(Q, V)$*, we define the* ***quotient space*** *with respect to a set* $S$ *of vectors in* $V$ *to be the subspace of* $V$ *orthogonal to every element in* $S$ *under* $Q$*.*

**Lemma 5.3.2** *Let* $(Q, V)$ *be a quadratic space with signature* $(1, n)$ *and* $B$ *be the bilinear form underlying* $Q$*. If* $\mathbf{x}$ *and* $\mathbf{y}$ *are isotropic vectors and* $B(\mathbf{x}, \mathbf{y}) = 0$*, then* $\mathbf{x} = C\mathbf{y}$ *for some non-zero constant* $C$*.*

*Proof:* Because $(Q, V)$ is a quadratic space of signature $(1, n)$, there exists an ordered basis $\beta = (\boldsymbol{\beta}_0, ..., \boldsymbol{\beta}_n)$ for $V$ such that $Q(\boldsymbol{\beta}_0) = 1$ and $Q(\boldsymbol{\beta}_i) = -1$ for $i \in \{1, ..., n\}$ and the elements of $\beta$ is orthogonal under $B$. The assumptions imply that every nonzero vector in the subspace $U = \text{span}\{\mathbf{x}, \mathbf{y}\}$ is isotropic. Hence $U$ intersects the subspace $W$ spanned by $\boldsymbol{\beta}_1, ...\boldsymbol{\beta}_n$ in the zero subspace. It follows that $\dim(U) = 1$. Thus $\mathbf{x} = C\mathbf{y}$ for some nonzero constant $C$. ■

**Corollary 5.3.3** *If $(Q, V)$ is a quadratic space with signature $(n, 1)$, $B$ is the bilinear form underlying $Q$, $\mathbf{x}$ and $\mathbf{y}$ are isotropic vectors and $B(\mathbf{x}, \mathbf{y}) = 0$, then $\mathbf{x} = C\mathbf{y}$ for some non-zero constant $C$.*

## 5.4 Counting subspaces in a quadratic kernel

We will need the following result to establish some properties about lines in projective space later.

**Theorem 5.4.1** *Let $(Q, \mathbb{R}^n)$ be a quadratic space and $B$ be the bilinear form underlying $Q$. There are either zero, one, two, or infinitely many distinct one-dimensional subspaces of $\mathbb{R}^n$ such that $Q(\mathbf{u}) = 0$ for every $\mathbf{u}$ contained in one of these subspaces.*

*Proof:* Let $\beta = (\boldsymbol{\beta}_0, ...\boldsymbol{\beta}_n)$ be a basis for $(Q, \mathbb{R}^n)$ constructed as in Definition 5.2.1. Let $(i, j)$ be the signature of $(Q, \mathbb{R}^n)$, and $k = n - i - j$ represent the number of vectors in $\beta$ that resolve to 0 under the quadratic form $Q$. Then one of the following must hold:

- $(Q, \mathbb{R}^n)$ is positive definite or negative definite,
- $(Q, \mathbb{R}^n)$ is of signature $(1, 1)$ with k=0,
- $(Q, \mathbb{R}^n)$ contains a subspace of signature $(1, 2)$ or $(2, 1)$,
- $(Q, \mathbb{R}^n)$ has $k = 1$ and is positive semi-definite or negative semi-definite,
- $(Q, \mathbb{R}^n)$ has $k > 1$ and is positive semi-definite or negative semi-definite, *or*

- $(Q, \mathbb{R}^n)$ has $k > 0$ and contains a subspace of signature $(1, 1)$.

If $(Q, \mathbb{R}^n)$ is positive definite or negative definite then it is anistropic, so we find no subspaces containing isotropic vectors.

If $(Q, \mathbb{R}^n)$ is of signature $(1, 1)$ and $k = 0$, then the subspaces spanned by $\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_0 - \boldsymbol{\beta}_1$ are distinct, and each nonzero vector contained in them is isotropic. Moreover, these are the only isotropic vectors in this space, so we find two distinct one-dimensional subspaces in the kernel of $Q$.

If $(Q, \mathbb{R}^n)$ contains a subspace of signature $(1, 2)$, which has an ordered basis $\gamma$ constructed as specified in Definition 5.2.1, then every vector in the subspace spanned by $\boldsymbol{\gamma}_0 + \alpha^2\boldsymbol{\gamma}_1 + \sqrt{1 - \alpha^2}\boldsymbol{\gamma}_2$ maps to 0 under our quadratic form for $\alpha \in [0, 1] \subseteq \mathbb{R}$. So there are infinitely many one-dimensional subspaces in the kernel of $Q$. Similarly, a subspace with signature $(2, 1)$ yields infinitely many one-dimensional subspaces in the kernel of $Q$.

If $k = 1$ and $(Q, \mathbb{R}^n)$ is semi-definite, then there is only one one-dimensional subspace, spanned by $\beta_n$ contained in the kernel of $Q$.

If $k > 0$ and $(Q, \mathbb{R}^n)$ is semi-definite, then any combination of the $k$ basis vectors that map to 0 under $Q$ yields a basis for a one-dimensional subspace of vectors in the kernel of $Q$, so there are infinitely many one-dimensional subspaces contained in the kernel of $Q$.

If $k > 0$ and $(Q, \mathbb{R}^n)$ contains a subspace of signature $(1, 1)$, then any combination of $\beta_n$ and an isotropic vector from that subspace is a basis for a one-dimensional subspace of vectors in the kernel, so there are infinitely many one-dimensional subspaces contained in the kernel of $Q$.

Collecting these results, we see that there are zero, one, two, or infinitely many one-dimensional linear subspaces contained in the kernel of $Q$. ■

# Chapter 6: Plücker Coordinates

## 6.1 Ray coordinates

**Definition 6.1.1** *If $P_{\mathbf{x}}$ and $P_{\mathbf{y}}$ are distinct (oriented) points then we say that the elements of the matrix $(\xi_{ij})$ are* ***Plücker ray coordinates*** *of the (oriented) line $P_{\mathbf{x}} \vee P_{\mathbf{y}}$ where*

$$\xi_{ij} := \begin{vmatrix} x_i & x_j \\ y_i & y_j \end{vmatrix}. \tag{6.1.1}$$

If $P_{\mathbf{x}'}$ and $P_{\mathbf{y}'}$ are (oriented) points such that $P_{\mathbf{x}'} \vee P_{\mathbf{y}'}$ has Plücker ray coordinates $(\xi'_{ij})$ and $P_{\mathbf{x}'} \vee P_{\mathbf{y}'} = P_{\mathbf{x}} \vee P_{\mathbf{y}}$, then there exists an invertible $2 \times 2$ matrix A (with positive determinant) such that $(x', y') = (x, y)A$. If follows that $\xi'_{ij} = \det(A)\xi_{ij}$. So Plücker ray coordinates are well defined up to a (positive) scalar multiple.

Note that $\xi_{ii} = 0$ and $\xi_{ij} = -\xi_{ji}$. While the above definition holds in an (oriented) projective space of arbitrary dimension, we are concerned with (oriented) lines in three-dimensional projective space. In this particular case we can uniquely specify all 16 terms of the matrix $(\xi_{ij})$ with only six terms by exploiting the fact that this matrix is skew-symmetric and is zero along its diagonal. This motivates consideration of the following vector, which uniquely specifies the $4 \times 4$ matrix $(\xi_{ij})$:

**Definition 6.1.2** *If* $(\xi_{ij})$ *are Plücker ray coordinates for an (oriented) line in* $\mathbf{OPG}(3, F)$ *we define* $\boldsymbol{\xi}$ *to be the column vector*

$$\boldsymbol{\xi} \coloneqq (\xi_{01}, \xi_{02}, \xi_{03}, \xi_{23}, \xi_{31}, \xi_{12}). \tag{6.1.2}$$

While this selection of terms will be justified later in Theorem 7.4.1, it can be noted that these terms are the six $2 \times 2$ minor determinants of the matrix

$$\begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{bmatrix}. \tag{6.1.3}$$

**Notation 6.1.3** *The oriented Plücker ray coordinates for the oriented line* $A \vee B$ *are a negative scalar multiple of any Plücker ray coordinates for the oriented line* $B \vee A$*, so we write* $A \vee B = -B \vee A$ *to indicate this relationship. It can be easily verified for distinct oriented points* $A$ *and* $B$ *with* $|A| \neq |B|$ *that*

$$-(A \vee B) = (-A) \vee B = A \vee (-B) = B \vee A, \tag{6.1.4}$$

*so this notation is not ambiguous.*

### 6.2 Axial coordinates

We now engage in a similar construction using the tangential coordinates of a pair of planes instead of the homogeneous coordinates of a pair of points. The same justification used above to show that the Plücker ray coordinates are well-defined holds here as well and so we do not repeat it.

**Definition 6.2.1** *If $\Pi_\pi$ and $\Pi_\psi$ are distinct (oriented) planes in* $\mathbf{OPG}(3, F)$, *we say that* $(\Xi_{ij})$ *are **Plücker axial coordinates** of the (oriented) line* $\Pi_\pi \wedge \Pi_\psi$, *where*

$$\Xi_{ij} := \begin{vmatrix} \pi_i & \pi_j \\ \psi_i & \psi_j \end{vmatrix}. \tag{6.2.1}$$

As above with the Plücker ray coordinates, we have $\Xi_{00} = \Xi_{11} = \Xi_{22} = \Xi_{33} = 0$ and $\Xi_{ij} = -\Xi_{ji}$, so for a three-dimensional projective space we can uniquely specify all 16 terms of the matrix $(\Xi_{ij})$ with just six terms.

**Definition 6.2.2** *If* $(\Xi_{ij})$ *are Plücker axial coordinates for an (oriented) line in* $\mathbf{OPG}(3, F)$ *we define* $\boldsymbol{\Xi}$ *to be the row vector*

$$\boldsymbol{\Xi} := [\Xi_{23}, \Xi_{31}, \Xi_{12}, \Xi_{01}, \Xi_{02}, \Xi_{03}]. \tag{6.2.2}$$

## 6.3 Line coordinates

**Theorem 6.3.1** *Let $\ell$ be a line with Plücker axial coordinates* $(\Xi_{ij})$ *and Plücker ray coordinates* $(\xi_{ij})$. *Then, the vectors* $\boldsymbol{\Xi}$ *and* $\boldsymbol{\xi}$ *satisfy*

$$\boldsymbol{\xi} = C\boldsymbol{\Xi}^T, \tag{6.3.1}$$

*where C is a non-zero scalar.*

*Proof:* A straightforward proof of this based on one on page 89 of Coxeter[3] follows. This proof has been extended for readability and modified to address the case where one or more of $\xi_{01}$, $\xi_{02}$, $\xi_{03}$, $\xi_{23}$, $\xi_{31}$, or $\xi_{12}$ equals zero, which occurs quite often in practice.

Let the oriented points $P_{\mathbf{x}}$ and $P_{\mathbf{y}}$ each be incident with both of the oriented planes $\Pi_{\boldsymbol{\pi}}$ and $\Pi_{\boldsymbol{\psi}}$. Then

$$\langle \mathbf{x}, \boldsymbol{\pi} \rangle = \langle \mathbf{y}, \boldsymbol{\pi} \rangle = \langle \mathbf{x}, \boldsymbol{\psi} \rangle = \langle \mathbf{y}, \boldsymbol{\psi} \rangle = 0.$$

Let $(\xi_{ij})$ be Plucker ray coordinates for the line $P_{\mathbf{x}} \vee P_{\mathbf{y}}$ and let $(\Xi_{ij})$ be Plucker axial coordinates for the line $\Pi_{\boldsymbol{\pi}} \wedge \Pi_{\boldsymbol{\psi}}$.

Consider

$$\begin{aligned} \sum_{k=0}^{3} \xi_{ik} \Xi_{jk} &= \sum_{k=0}^{3} (x_i y_k - y_i x_k) \cdot (\pi_j \psi_k - \psi_j \pi_k) \\ &= x_i \pi_j \langle \mathbf{y}, \boldsymbol{\psi} \rangle - x_i \psi_j \langle \mathbf{y}, \boldsymbol{\pi} \rangle - y_i \pi_j \langle \mathbf{x}, \boldsymbol{\psi} \rangle + y_i \psi_j \langle \mathbf{x}, \boldsymbol{\pi} \rangle \\ &= 0. \end{aligned} \tag{6.3.2}$$

Let $i, j$ be distinct elements of $\{0, \ldots, 3\}$. Then because $\xi_{ii} = \Xi_{ii} = 0$, two of the terms of $\xi_{i0}\Xi_{j0} + \cdots + \xi_{i3}\Xi_{j3}$ must equal 0. Therefore if we let $k_1$ and $k_2$ be the two values in $\{0, \ldots, 3\}$ not assumed by either $i$ or $j$, we obtain an equation of the form $\xi_{ik_1}\Xi_{jk_1} + \xi_{ik_2}\Xi_{jk_2} = 0$.

There are twelve ways to choose $i$ and $j$ in this fashion, so we obtain a system of twelve equations, which can be verified to be equivalent to the following equation with $C$ a nonzero constant:

$$\boldsymbol{\xi} = C \boldsymbol{\Xi}^T. \tag{6.3.3}$$

■

This affords us the luxury of using $\boldsymbol{\xi}$ and $\boldsymbol{\Xi}$ for a line in $\mathbf{PG}(3, F)$ interchangeably as long as we are not concerned with the orientation of the line.

**Definition 6.3.2** *If* $(\boldsymbol{\xi}_{ij})$ *are Plücker ray coordinates for the line* $\ell$ *or* $(\boldsymbol{\xi}_{ij})$ *are Plücker axial coordinates for the line* $\ell$ *then we say that* $(\boldsymbol{\xi}_{ij})$ *are* ***Plücker line coordinates*** *for the line* $\ell$*.*

## 6.4 Generalized Plücker coordinates

Plücker line coordinates are a specific case of a more general construction, the Plücker coordinates for a $k$-flat in an $n$-dimensional projective space.

We define a positively oriented basis as one that can be constructed from the standard basis for our space by pre-multiplying the basis vectors by a matrix that has a positive determinant.

Per Stolfi,[16] given a positively oriented ordered basis $\beta = (\boldsymbol{\beta}_0, ..., \boldsymbol{\beta}_n)$ for the underlying vector space, and a set of $k$ linearly independent vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k$ in our flat, the Plücker coordinates for the flat are uniquely specified by the $\binom{n+1}{k+1}$ minor determinants of order $k+1$ of the $(k+1) \times (n+1)$ matrix

$$\begin{bmatrix} v_0^0 & \cdots & v_n^0 \\ \vdots & \ddots & \vdots \\ v_0^k & \cdots & v_n^k \end{bmatrix}, \tag{6.4.1}$$

where $v_j^i$ refers to the the $j$-th component of $[\mathbf{v}^i]_\beta$.

In general it takes $\binom{n+1}{k+1}$ coefficients to describe a $k$-flat in an $n$-dimensional projective space using Plücker coordinates, and it takes $(n+1)(k+1)$ coordinates to define a flat using the coordinates of basis vectors for the flat, or $(n+1)(n-k)$ tangential coordinates required to describe the flat in terms of the subspace of $V$ orthogonal to it.

Consider Table 6.1, which describes the number of coordinates required to represent a $k$-flat in $n$ dimensions, in Plücker, homogeneous, and tangential form. As can be seen in

**Plücker**

| $k \backslash n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 0 | 3 | 4 | 5 | 6 | 7 |
| 1 | 3 | **6** | 10 | 15 | 21 |
| 2 | - | 4 | 10 | 20 | 21 |
| 3 | - | - | 5 | 15 | 35 |
| 4 | - | - | - | 6 | 21 |

**Homogeneous**

| $k \backslash n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 0 | 3 | 4 | 5 | 6 | 7 |
| 1 | 6 | 8 | 10 | **12** | **14** |
| 2 | - | 12 | 15 | 18 | 21 |
| 3 | - | - | 20 | 24 | 28 |
| 4 | - | - | - | 30 | 35 |

**Tangential**

| $k \backslash n$ | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|
| 0 | 6 | 12 | 20 | 30 | 42 |
| 1 | 3 | 8 | 15 | 24 | 35 |
| 2 | - | 4 | 10 | 18 | 28 |
| 3 | - | - | 5 | **12** | **21** |
| 4 | - | - | - | 6 | 14 |

Table 6.1. Number of coordinates required for the Plücker, homogeneous and tangential representations of a $k$-flat in $\mathbf{PG}(n, F)$

the table, the number of Plücker coordinates required to distinguish a line are only fewer than the number of homogeneous or tangential coordinates for lines (1-flats) in three dimensions. The coordinate format with strictly fewest coordinates for a given combination of $k$ and $n$ is given in bold to highlight the distinction. After that the binomial growth in the number of Plücker coordinates required outstrips the number required to describe the flat in terms of homogeneous or tangential coordinates.

One might consider this to be why Plücker coordinates seem to have been relegated to a mathematical back-water, because while they generalize to higher dimensional spaces, they become increasingly harder to work with and less and less useful as the dimension of our space increases.

This provides justification for our choice of representation for geometric primitives in $\mathbf{PG}(3, F)$ and $\mathbf{OPG}(3, F)$; our choice of homogeneous coordinates for points, Plücker coordinates for lines, and tangential coordinates for planes involves the fewest number of coordinates in each case, and we prefer homogeneous and tangential coordinates for points and planes because the standard actions require fewer operations in these cases than their Plücker equivalents.

# Chapter 7: Working with Plücker Coordinates

If Plücker line coordinates only saved us two coefficients in our line representation for the highly specific case of lines in a three-dimensional (oriented) projective space, then there probably would not be much interest in them. Fortunately, this is not the case; there are a number of computational benefits for working directly with Plücker line coordinates.

## 7.1 The Plücker quadric

**Definition 7.1.1** *Let $B : F^6 \times F^6 \to F$ be the symmetric bilinear form*

$$B(\mathbf{v}, \mathbf{w}) \coloneqq v_0 w_3 + v_1 w_4 + v_2 w_5 + w_0 v_3 + w_1 v_4 + w_2 v_5. \tag{7.1.1}$$

**Definition 7.1.2** *Let $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}$ be vectors in $F^4$. We define*

$$D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) \coloneqq \begin{vmatrix} p_0 & p_1 & p_2 & p_3 \\ q_0 & q_1 & q_2 & q_3 \\ r_0 & r_1 & r_2 & r_3 \\ s_0 & s_1 & s_2 & s_3 \end{vmatrix}. \tag{7.1.2}$$

**Theorem 7.1.3** *If (oriented) lines $P_{\mathbf{p}} \vee P_{\mathbf{q}}$ and $P_{\mathbf{r}} \vee P_{\mathbf{s}}$ in $\mathbf{OPG}(3, F)$ have Plücker ray coordinates $(\xi_{ij})$ and $(\xi'_{ij})$ respectively, then $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ is a nonzero (positive) scalar* *multiple of* $B(\boldsymbol{\xi}, \boldsymbol{\xi}')$.

*Proof:* Let $(\xi_{ij})$ and $(\xi'_{ij})$ be Plücker ray coordinates for $P_{\mathbf{p}} \vee P_{\mathbf{q}}$ and $P_{\mathbf{r}} \vee P_{\mathbf{s}}$ respectively. Then as noted by Pellegrini,[11] if we expand the determinant $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ according to the $2 \times 2$ minors of the submatrix formed by $\mathbf{p}$ and $\mathbf{q}$ and the minors of the submatrix formed by $\mathbf{r}$ and $\mathbf{s}$, we obtain

$$D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) = \xi_{01}\xi'_{23} + \xi_{02}\xi'_{31} + \xi_{03}\xi'_{12} + \xi'_{01}\xi_{23} + \xi'_{02}\xi_{31} + \xi'_{03}\xi_{12} = B(\boldsymbol{\xi}, \boldsymbol{\xi}'), \tag{7.1.3}$$

with $(\xi_{ij})$ and $(\xi'_{ij})$ determined up to a nonzero (positive) scalar multiple. ■

**Corollary 7.1.4** *For any line in* $\mathbf{PG}(3, F)$ *with Plücker line coordinates* $(\xi_{ij})$, $B(\boldsymbol{\xi},\boldsymbol{\xi})=0$.

*Proof:* Let $P_{\mathbf{p}} \vee P_{\mathbf{q}}$ be a line in $\mathbf{PG}(3, F)$ with Plücker line coordinates $(\xi_{ij})$. Then $B(\boldsymbol{\xi}, \boldsymbol{\xi}) = D(\mathbf{p}, \mathbf{q}, \mathbf{p}, \mathbf{q}) = 0$. ■

**Definition 7.1.5** *We call the quadratic form* $Q : F^6 \to F$ *the* ***Plücker quadratic form****, where*

$$Q(\mathbf{v}) := v_0 v_3 + v_1 v_4 + v_2 v_5. \tag{7.1.4}$$

It is readily apparent that $B(\mathbf{v}, \mathbf{v}) = 2Q(\mathbf{v})$.

**Lemma 7.1.6** *The quadratic space* $(Q, F^6)$ *is isotropic.*

*Proof:* Let $\ell$ be a line in $PG(3, F)$ and let $(\xi_{ij})$ be Plücker line coordinates for $\ell$, then $Q(\boldsymbol{\xi}) = \frac{1}{2}B(\boldsymbol{\xi}, \boldsymbol{\xi}) = 0$. Since $\boldsymbol{\xi} \neq \mathbf{0}$, the quadratic space $(Q, F^6)$ contains a nonzero vector in its kernel and so $(Q, F^6)$ is isotropic. ■

**Definition 7.1.7** *In* $\mathbf{PG}(5, F)$, *we define the Plücker quadric* $\mathcal{Q}$ *to be*

$$\mathcal{Q} := \{p(\ell) : \ell \text{ is an line in } \mathbf{PG}(3, F)\}. \tag{7.1.5}$$

### 7.2 Line interactions in **OPG**$(3, F)$

**Theorem 7.2.1** *Let $\ell$ and $\ell'$ be lines in* $\mathbf{PG}(3, F)$ *with Plücker coordinates* $(\xi_{ij})$ *and* $(\xi'_{ij})$ *respectively. Then $\ell$ and $\ell'$ intersect if and only if* $B(\boldsymbol{\xi}, \boldsymbol{\xi}') = 0$.

*Proof:* Let $P_{\mathbf{p}}$, $P_{\mathbf{q}}$, $P_{\mathbf{r}}$, and $P_{\mathbf{s}}$ be points such that $\ell = P_{\mathbf{p}} \vee P_{\mathbf{q}}$ and $\ell' = P_{\mathbf{r}} \vee P_{\mathbf{s}}$. Then $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) = B(\boldsymbol{\xi}, \boldsymbol{\xi}')$ by Theorem 7.1.3. The only way $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) = 0$ is when these four vectors are not linearly independent. Since $\{\mathbf{p}, \mathbf{q}\}$ is linearly independent and $\{\mathbf{r}, \mathbf{s}\}$ is linearly independent, the only way this can occur is for the two lines to share a common subspace and hence intersect at least at one point. ■

**Definition 7.2.2** *Let $\ell$ and $\ell'$ be oriented lines in* $\mathbf{OPG}(3, F)$ *with Plücker ray coordinates* $(\xi_{ij})$ *and* $(\xi'_{ij})$ *respectively. We define the side operation $\Diamond$ as*

$$\ell \Diamond \ell' := \operatorname{sgn}(B(\boldsymbol{\xi}, \boldsymbol{\xi}')), \tag{7.2.1}$$

*where* sgn *is defined as*

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases} \tag{7.2.2}$$

Since $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are defined up to a positive scalar multiple, $B(\boldsymbol{\xi}, \boldsymbol{\xi}')$ is also, and so its sign is well-defined.

**Theorem 7.2.3** *Let $\ell = P_{\mathbf{p}} \vee P_{\mathbf{q}}$ and $\ell' = P_{\mathbf{r}} \vee P_{\mathbf{s}}$ be oriented lines in* $\mathbf{OPG}(3, F)$. *Then*

$$\ell \Diamond \ell' = \operatorname{sgn}(D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})). \tag{7.2.3}$$

*Proof:* Let $(\xi_{ij})$ and $(\xi'_{ij})$ be Plücker ray coordinates for $\ell$ and $\ell'$ respectively. We know that $\ell \Diamond \ell' = \mathrm{sgn}(B(\boldsymbol{\xi}, \boldsymbol{\xi}')) = \mathrm{sgn}(CD(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}))$, where C is a positive scalar factor, so $\ell \Diamond \ell' = \mathrm{sgn}(D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}))$. ■

The value of $(P_\mathbf{p} \vee P_\mathbf{q}) \Diamond (P_\mathbf{r} \vee P_\mathbf{s})$ admits an intuitive interpretation in terms of the "right hand rule." The value of $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s})$ is proportional to the signed volume of the tetrahedron formed by the vertices $\mathbf{p}, \mathbf{q}, \mathbf{r}$, and $\mathbf{s}$. The sign of this area can only be positive if $P_\mathbf{p} \vee P_\mathbf{q}$ does not intersect $P_\mathbf{r} \vee P_\mathbf{s}$. Note that if they are parallel they still intersect at a point at infinity.

Another way to think about it when both lines are affine lines and are not parallel is to "look down" from $P_\mathbf{s}$ towards the plane with normal $\mathbf{s} - \mathbf{r}$ that passes through the point $P_\mathbf{r}$, and consider the projection of $P_\mathbf{p} \vee P_\mathbf{q}$ onto that plane. For $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) > 0$ we see a ray heading from one point to another in a direction that is oriented counterclockwise around the $P_\mathbf{r}$. Similarly for $D(\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s}) < 0$ we see a vector based at the projection of $P_\mathbf{r}$ onto this plane, pointed in a direction with clockwise orientation around the point $P_\mathbf{r}$. Finally, if the lines intersect we see a ray that passes through the point $P_\mathbf{r}$.

### 7.3 Mapping lines onto points and hyperplanes

**Definition 7.3.1** *We define a mapping $p : \ell \mapsto p(\ell)$, which maps (oriented) lines in* $\mathbf{OPG}(3, F)$ *to (oriented) points in* $\mathbf{OPG}(5, F)$ *as follows. Let $\ell$ be an (oriented) line with Plücker ray coordinates $(\xi_{ij})$, then*

$$p(\ell) \coloneqq P_{\boldsymbol{\xi}}\,, \tag{7.3.1}$$

*where $P_{\boldsymbol{\xi}}$ refers to the (oriented) point specified by the vector $\boldsymbol{\xi}$.*

**Definition 7.3.2** *We define a mapping $\pi : \ell \mapsto \pi(\ell)$, which maps (oriented) lines in* $\mathbf{OPG}(3, F)$ *to (oriented) hyperplanes in* $\mathbf{OPG}(5, F)$ *as follows. Let $\ell$ be an (oriented) line* *with Plücker ray coordinates* $(\xi_{ij})$*, then* $\pi(\ell)$ *returns the hyperplane with the tangential coordinates* $\boldsymbol{\xi}^T[B]_\beta$.

**Corollary 7.3.3** *Let* $\ell$ *and* $\ell'$ *be oriented lines in* $\mathbf{OPG}(3, F)$*. If* $\Pi_{\boldsymbol{\pi}} = \pi(\ell')$ *and* $P_{\mathbf{v}} = p(\ell)$ *then we can compute* $\Diamond$ *using*

$$\ell \Diamond \ell' = \operatorname{sgn}(\langle \boldsymbol{\pi}, \mathbf{v} \rangle). \tag{7.3.2}$$

*Proof:* This is an immediate consequence of the definitions of the above mappings and the definition for $\Diamond$. ■

### 7.4 Visualizing Plücker coordinates

The following theorem helps motivate the seemingly strange choice of $\xi_{31}$ as one of the components of $\boldsymbol{\xi}$.

**Theorem 7.4.1** *Let* $\ell$ *be an (oriented) affine line with Plücker ray coordinates* $(\xi_{ij})$*. Then there exists a pair of (oriented) affine points* $P_{\mathbf{p}}$ *and* $P_{\mathbf{q}}$ *on* $\ell$ *with affine coordinates* $\widetilde{\mathbf{p}}$ *and* $\widetilde{\mathbf{q}}$ *respectively such that* $\ell = P_{\mathbf{p}} \vee P_{\mathbf{q}}$ *and*

$$\boldsymbol{\xi} = (\widetilde{\mathbf{q}} - \widetilde{\mathbf{p}}, \widetilde{\mathbf{p}} \times \widetilde{\mathbf{q}}). \tag{7.4.1}$$

*Proof:* The first three terms are defined in terms of a determinant in which each term is multiplied by the same factor you divide by to obtain the affine coordinate, so the first three Plücker coordinates are the same as the difference of the affine coordinates. The cross product is obtained because the three remaining terms are the just three minor determinants of the affine coordinates taken in the same order as their cross product, and $\xi_{31} = -\xi_{13}$, so the sign flip of the second term of the cross product is already taken into account. This mnemonic device is the reason why $\xi_{31}$ is traditionally selected instead of $\xi_{13}$ for the fifth coefficient of $\boldsymbol{\xi}$. ■

One can recognize that a line is a line at infinity by noting the following:

**Theorem 7.4.2** *If $\ell$ is a line at infinity with Plücker ray coordinates $(\xi_{ij})$ then $\boldsymbol{\xi}$ is of the form $(0, 0, 0, \xi_{23}, \xi_{31}, \xi_{12})$ and these are the only Plücker coordinates of this form.*

*Proof (⇒):* A line at infinity is a join of two points at infinity. So for any two points on the line, the first components of their homogeneous coordinates **a** and **b** must equal 0. Hence $\xi_{0i} = a_0 b_i - b_0 a_i = 0$ for $i \in \{0, ..., 3\}$ and so the first three components, $\xi_{01}$, $\xi_{02}$, and $\xi_{03}$, contained in $\boldsymbol{\xi}$ equal 0. ■

*Proof (⇐):* Conversely, assume $\ell$ is not a line at infinity. Then it has a basis $\{\mathbf{x}, \mathbf{y}\}$, where $P_{\mathbf{x}}$ and $P_{\mathbf{y}}$ are not points at infinity. So these points have affine coordinates $\widetilde{\mathbf{x}}$ and $\widetilde{\mathbf{y}}$ respectively. But since $\widetilde{x}_i - \widetilde{y}_i = 0$ for $i = 1, 2, 3$, we know $\widetilde{\mathbf{x}} = \widetilde{\mathbf{y}}$ and that $\mathbf{x}$ is a scalar multiple of $\mathbf{y}$ contradicting that they form a basis for $\ell$. ■

This observation gives rise to a natural parametrization of the line given by its Plücker coordinates. The following parametrization is taken from Duguet.[4]

**Definition 7.4.3** *For an affine line with Plücker coordinates $(\xi_{ij})$, where $\boldsymbol{\xi} = (\mathbf{u}, \mathbf{v})$ we define $M_{\boldsymbol{\xi}} : F \to F^3$ as*

$$M_{\boldsymbol{\xi}}(\lambda) \coloneqq \frac{\mathbf{u} \times \mathbf{v}}{\langle \mathbf{u}, \mathbf{u} \rangle} + \lambda \frac{\mathbf{u}}{||\mathbf{u}||}. \tag{7.4.2}$$

The above parametrization yields affine coordinates for all affine points contained in $\ell$. The starting point $M_{\boldsymbol{\xi}}(0)$ is the point on the line nearest the origin, and this parametrization has unit speed. If $(\xi_{ij})$ are Plücker ray coordinates for an oriented line $\ell = A \vee B$, then $\ell$ is parametrized in the direction from A to B.

Since Plücker coordinates are defined up to a (positive) scalar multiple, we can scale the Plücker coordinates $(\xi_{ij})$, to obtain other Plücker coordinates $(\xi'_{ij})$ for the same oriented line such that $\boldsymbol{\xi}' = (\mathbf{u}, \mathbf{v})$ and

$$M_{\boldsymbol{\xi}'}(\lambda) \coloneqq \mathbf{u} \times \mathbf{v} + \lambda \mathbf{u}. \tag{7.4.3}$$

# Chapter 8: Study Coordinates

## 8.1 Introducing Study coordinates

We have already established that Plücker coordinates are useful for many things, but there is a basis for $F^6$ that diagonalizes our quadric and has other nice properties that merit consideration.

**Definition 8.1.1** *Let $\gamma$ be the ordered basis for $F^6$ given in terms of the standard basis $\beta = \{e_1, ..., e_6\}$ as follows:*

$$\gamma = \left( \frac{e_1 + e_4}{\sqrt{2}}, \frac{e_2 + e_5}{\sqrt{2}}, \frac{e_3 + e_6}{\sqrt{2}}, \frac{e_1 - e_4}{\sqrt{2}}, \frac{e_2 - e_5}{\sqrt{2}}, \frac{e_3 - e_6}{\sqrt{2}} \right). \tag{8.1.1}$$

Because we are generally only concerned with whether or not our oriented line is positively or negatively oriented and are likely to normalize the result, we can generally ignore the positive constant factor $\frac{1}{\sqrt{2}}$ in the basis for computational purposes. The square of the change of coordinate matrix $[I]_\beta^\gamma$ is the identity matrix, so we can go back and forth between bases by adding and subtracting terms as indicated, but if we drop the constant factor then we return to a different set of Plücker line coordinates for the same (oriented) line when we apply it a second time.

We motivate this choice of basis by the observation that under this basis, the matrix representation for the bilinear form underlying the Plücker quadric (given by

Definition 7.1.1) is diagonalized to the following form:

$$[B]_\gamma := \begin{bmatrix} I_3 & 0 \\ 0 & -I_3 \end{bmatrix}. \tag{8.1.2}$$

This reveals that the quadratic space $(Q, F^6)$ is of signature $(3,3)$, which was obscured when we were looking at it under the standard basis for $F^6$.

**Definition 8.1.2** *If $(\xi_{ij})$ are Plücker ray coordinates for the oriented line $\ell$, then we refer to the column vector $[\boldsymbol{\xi}]_\gamma$ as **Study coordinates** for $\ell$.*

### 8.2 The Study representative pair

To further motivate this choice of basis as a means to describe a line, we make the following observation.

If we fix an arbitrary point $P$ in $\mathbf{PG}(3, F)$ and then consider a line $A \vee B$, then the lines $A \vee P$ and $A \vee B$ intersect the hyperplane $P^\Delta$ at a pair of "representative" points, which can be used to describe $A \vee B$.[3] If $A \vee B$ contains $P$ then both representative points are the same point.

In particular, we have already singled out the vector $\boldsymbol{\beta}_0$ taken from the standard basis $(\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \boldsymbol{\beta}_3)$ of $F^4$ as orthogonal to our hyperplane at infinity, so we choose to recycle it here. That way both of these "representative" points are contained in the hyperplane at infinity, and so both of them are points at infinity.

**Definition 8.2.1** *We define the **Study representative pair** $\{P_{\mathbf{u}}; P_{\mathbf{v}}\}$ in terms of the Study coordinates for an (oriented) line to be the (oriented) points at infinity with the* *homogeneous coordinates* **u** *and* **v**, *where*

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} := [\boldsymbol{\xi}]_\gamma, \quad \mathbf{u} := \begin{bmatrix} 0 \\ u_1 \\ u_2 \\ u_3 \end{bmatrix}, \quad \mathbf{v} := \begin{bmatrix} 0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}. \tag{8.2.1}$$

It can be shown by means of Clifford transformations that the Study representative pair is precisely the representative pair formed when we choose the point containing $\boldsymbol{\beta}_0$ as the arbitrary fixed point and represent the line $X \vee Y$ by intersecting the lines $X \vee P_{\boldsymbol{\beta}_0}$ and $P_{\boldsymbol{\beta}_0} \vee Y$ with the hyperplane at infinity. For proof of this see Coxeter.[3] This motivates the term Study representative pair and the choice to represent **u** and **v** as points at infinity.

# Chapter 9: Working with Study Coordinates

## 9.1 Lines at infinity

**Theorem 9.1.1** *A line $\ell$ is a line at infinity if and only if its Study representative pair is of the form* $\{U; -U\}$.

*Proof:* ($\Rightarrow$) Recall that the Plücker coordinates $(\xi_{ij})$ of a line at infinity are all of the form $\boldsymbol{\xi} = (0, \mathbf{v})$. So the associated Study representative pair must be $\{P_{\mathbf{v}}; P_{-\mathbf{v}}\} = \{P_{\mathbf{v}}; -P_{\mathbf{v}}\}$.

($\Leftarrow$) Assume that the Study representative pair is of this form. The Plücker coordinates $(\xi_{ij})$ for a line with Study representative pair $\{P_{\mathbf{u}}; -P_{\mathbf{u}}\}$ can be found by changing basis from our Study representative pair. The coordinates of such a line are $\boldsymbol{\xi} = (\mathbf{u} - \mathbf{u}, \mathbf{u} + \mathbf{u}) = (0, \mathbf{u})$, so $\{P_{\mathbf{u}}; -P_{\mathbf{u}}\}$ represents a line at infinity. ■

## 9.2 Line interactions in Study coordinates

Changing basis does not affect the relative position of points and hyperplanes, so we can compute the side operation directly in Study coordinates:

**Theorem 9.2.1** *If $\ell$ and $\ell'$ are oriented lines with Study ray coordinates $[\boldsymbol{\xi}]_\gamma$ and $[\boldsymbol{\xi}']_\gamma$ respectively, then*

$$\ell \Diamond \ell' = \operatorname{sgn}([\boldsymbol{\xi}]_\gamma^T [B]_\gamma [\boldsymbol{\xi}']_\gamma). \tag{9.2.1}$$

*Proof:* The matrix representation of the bilinear form $B$ taken from Equation 7.1.1 above is

$$B(\boldsymbol{\xi}, \boldsymbol{\xi}') = \boldsymbol{\xi}^T \, [B]_\beta \, \boldsymbol{\xi}', \tag{9.2.2}$$

where $[B]_\beta$ is the matrix form of $B$ relative to the standard basis $\beta$. Rewriting this in terms of the new basis, we obtain

$$\begin{aligned} B(\boldsymbol{\xi}, \boldsymbol{\xi}') &= \boldsymbol{\xi}^T \, [B]_\beta \, \boldsymbol{\xi}' \\ &= \boldsymbol{\xi}^T [I]_\gamma^\beta [B]_\gamma [I]_\beta^\gamma \boldsymbol{\xi}' \\ &= ([I]_\beta^\gamma \boldsymbol{\xi})^T [B]_\gamma [\boldsymbol{\xi}']_\gamma. \\ &= ([\boldsymbol{\xi}]_\gamma)^T [B]_\gamma [\boldsymbol{\xi}']_\gamma \\ &= [\boldsymbol{\xi}]_\gamma^T [B]_\gamma [\boldsymbol{\xi}']_\gamma. \end{aligned} \tag{9.2.3}$$

Substitute this result in the definition for $\Diamond$. ■

We use this to establish yet another way to calculate $\ell \Diamond \ell'$:

**Corollary 9.2.2** *Given a pair of oriented lines $\ell$ and $\ell'$ with Study line coordinates $[\boldsymbol{\xi}]_\gamma = (\mathbf{p}, \mathbf{q})$ and $[\boldsymbol{\xi}']_\gamma = (\mathbf{r}, \mathbf{s})$ respectively where $\mathbf{p}, \mathbf{q}, \mathbf{r}, \mathbf{s} \in F^3$ their relative orientation $\ell \Diamond \ell'$ can be calculated by*

$$\ell \Diamond \ell' = \operatorname{sgn}(\langle \mathbf{p}, \mathbf{r} \rangle - \langle \mathbf{q}, \mathbf{s} \rangle). \tag{9.2.4}$$

# Chapter 10: Arrangements and Raycasting

## 10.1 Arrangements

We borrow the following definition from Halperin.[8]

**Definition 10.1.1** *Let $\mathcal{H}$ be a finite set of hyperplanes in $\mathbb{R}^d$. Then the hyperplanes in $\mathcal{H}$ induce a decomposition of $\mathbb{R}^d$ into connected open cells called the* ***arrangement*** *$A(\mathcal{H})$. A* ***cell*** *in $A(\mathcal{H})$ is a maximal connected region of $\mathbb{R}^d$ not intersected by any hyperplane in $\mathcal{H}$.*

**Definition 10.1.2** *We define a* ***closed cell*** *$C'$ to be the expansion of a cell $C$ to include all of its boundary points $\partial C$. We call $C'$ the* ***closure*** *of $C$.*

Every closed cell associated with an arrangement $A(\mathcal{H})$ can be constructed by intersecting halfspaces associated with the hyperplanes in $A(\mathcal{H})$ by choosing for every $\Pi \in H$, one of $H^+_\Pi$ or $H^-_\Pi$ and then intersecting the selected halfspaces. Because a closed cell contains all of its boundary points, it is a closed set.

**Definition 10.1.3** *Given a finite set of triangles $T$ in* $\mathbf{PG}(3, \mathbb{R})$*, we obtain a set of hyperplanes in $\mathbb{R}^6$ by setting*

$$\mathcal{H}_T = \{\pi(\ell) : \ell \textit{ is a line that contains an edge of a triangle in } T\}. \qquad (10.1.1)$$

**Definition 10.1.4** *We refer to the open set of points in a plane that is bounded by a set of three distinct pairwise incident lines as an* ***open triangle****.*

**Definition 10.1.5** *Given an open triangle $\Delta$ we define a* ***closed triangle*** *$\Delta'$ to be the expansion of $\Delta$ to include all of its boundary points $\partial\Delta$ in the plane, and we call $\Delta'$ the* ***closure*** *of $\Delta$.*

**Claim 10.1.6** *Let $\ell$ be an oriented line and let A, B, and C be positively oriented points. Let $\ell_1 = A \vee B$, $\ell_2 = B \vee C$, and $\ell_3 = C \vee A$. Let $\Delta$ be the open triangle formed by $|\ell_1|, |\ell_2|, |\ell_3|$. Then $\ell$ intersects $\Delta$ if and only if*

- $\ell \Diamond \ell_i > 0$ *holds for all* $i \in \{1, 2, 3\}$ *or*
- $\ell \Diamond \ell_i < 0$ *holds for all* $i \in \{1, 2, 3\}$.

**Claim 10.1.7** *Let $\ell$ be an oriented line and let A, B, and C be positively oriented points. Let $\ell_1 = A \vee B$, $\ell_2 = B \vee C$, and $\ell_3 = C \vee A$. Let $\Delta'$ be the closed triangle formed by $|\ell_1|, |\ell_2|, |\ell_3|$. Then $\ell$ intersects $\Delta'$ if and only if*

- $\ell \Diamond \ell_i \geq 0$ *holds for all* $i \in \{1, 2, 3\}$ *or*
- $\ell \Diamond \ell_i \leq 0$ *holds for all* $i \in \{1, 2, 3\}$.

**Definition 10.1.8** *Given a set of planar points, $P_1, ..., P_n$ specified in either clockwise or counterclockwise order on a plane we define the* ***open polygon*** *$\Delta(P_1, ..., P_n)$ be the open set of points bounded by the lines $P_i \vee P_{i+1}$ and $P_n \vee P_1$ where $i \in \{1, ..., n-1\}$.*

**Definition 10.1.9** *Similarly we refer to the expansion of $\Delta(P_1, ..., P_n)$ to include its boundary points in the plane as the* ***closed polygon*** *$\Delta'(P_1, ..., P_n)$ and call $\Delta'(P_1, ...P_n)$ the* ***closure*** *of $\Delta(P_1, ..., P_n)$.*

**Definition 10.1.10** *Given a set of planar points, $P_1, ..., P_n$ specified in counterclockwise order on an oriented plane we define the* ***oriented polygon*** $\Delta(P_1, ..., P_n)$ *to be the set of oriented points bounded by the oriented lines $P_i \vee P_{i+1}$ and $P_n \vee P_1$ where $i \in \{1, ..., n-1\}$ and we take the orientation of our oriented polygon from the orientation of our plane. We define* ***closed oriented polygons*** *and* ***open oriented polygons*** *based on whether or not we include the boundary points as above, and refer to* ***open*** *and* ***closed oriented triangles*** *by a similar construction.*

Given the same points in clockwise order, we obtain an oriented polygon with the same boundary, but opposite orientation.

**Theorem 10.1.11** *Let $T$ be a finite set of open triangles in* $\mathbf{PG}(3, \mathbb{R})$ *and $\ell_1$ and $\ell_2$ be oriented lines in* $\mathbf{OPG}(3, \mathbb{R})$*. Let $C \in A(\mathcal{H}_T)$. If $p(\ell_1) \in C$ and $p(\ell_2) \in C$ then $|\ell_1|$, and $|\ell_2|$ intersect the same open triangles in $T$.*

*Proof:* Assume that $p(\ell_1) \in C$, $p(\ell_2) \in C$ and that $\ell_1$ intersects the open triangle $\Delta$ in $T$. Let the oriented lines $\Delta_1$, $\Delta_2$, $\Delta_3$ constructed as in Claim 10.1.7 be the edges of $\Delta$. Because $\ell_1$ intersects $\Delta$ we know that either $\ell_1 \Diamond \Delta_i > 0$ holds for $i = 1, 2, 3$ or $\ell_1 \Diamond \Delta_i < 0$ must hold for $i$=1,2,3. Without loss of generality, assume $\ell_1 \Diamond \Delta_i > 0$. Yet $p(\ell_2)$ lies in the same set of open half-spaces as $p(\ell_1)$, so $\ell_2 \Diamond \Delta_i > 0$ holds for $i \in \{1, 2, 3\}$. Therefore $\ell_2$ also intersects $\Delta$. See Pellegrini,[10] p.8 for an alternate proof. ■

**Corollary 10.1.12** *Given a finite set of open triangles $T$ in* $\mathbf{PG}(3, \mathbb{R})$*, if $p(\ell_1)$ is in a cell $C$ of $A(\mathcal{H}_T)$ and $p(\ell_2)$ is contained in the closure of $C$ and $|\ell_1|$ intersects an open triangle $\Delta$, then $|\ell_2|$ intersects the closure of $\Delta$.*

We can extend both of these results to convex polygons by noting that any convex polygon can be constructed by intersecting a number of triangles in a plane. Similarly they can be extended to nonconvex polygons by first subdividing the polygon into convex regions. This yields the following definition:

**Definition 10.1.13** *Given a finite set of polygons $T$ in* $\mathbf{PG}(3,\mathbb{R})$*, we obtain a set of hyperplanes in* $\mathbb{R}^6$ *by setting*

$$\mathcal{H}_P = \{\pi(\ell) : \ell \text{ is a line that contains an edge of a polygon in } T\}. \tag{10.1.2}$$

With this definition, a minor generalization yields the following two corollaries for Theorem 10.1.11.

**Corollary 10.1.14** *Let $\mathcal{P}$ be a finite set of open polygons in* $\mathbf{PG}(3,\mathbb{R})$ *and $\ell_1$ and $\ell_2$ be oriented lines in* $\mathbf{OPG}(3,\mathbb{R})$*. Let $C \in A(\mathcal{H}_{\mathcal{P}})$. If $p(\ell_1) \in C$ and $p(\ell_2) \in C$ then $|\ell_1|$, and $|\ell_2|$ intersect the same open polygons in $\mathcal{P}$.*

**Corollary 10.1.15** *Given a finite set of open polygons $\mathcal{P}$ in* $\mathbf{PG}(3,\mathbb{R})$*, if $p(\ell_1)$ is in a cell $C$ of $A(\mathcal{H}_{\mathcal{P}})$ and $p(\ell_2)$ is contained in the closure of $C$ and $|\ell_1|$ intersects an open polygon $\Delta$ in $\mathcal{P}$, then $|\ell_2|$ intersects the closure of $\Delta$.*

## 10.2 Raycasting

Assume that we have a three-dimensional scene constructed with a number of closed convex planar polygons. In this context, the problem of raycasting concerns itself with finding polygon intersections along a particular line.

For the purposes of this section, we will assume that the scene has been partitioned into convex polyhedra whose interiors are pairwise disjoint, such that all polygons in the scene are faces shared by these polyhedra. Furthermore, we assume that we can construct a graph (the **scene graph**) with polyhedra as nodes and the faces that do not represent polygons in our original scene as edges. We refer to these faces as **portals**. There are a number of algorithms in computational geometry to construct this data structure; the most obvious one is to use a binary space partitioning tree (BSP tree). Finally we will assume that the edges of the faces can all be retrieved in counterclockwise order, from the perspective of outside the polyhedron, so for any given node in the scene graph, we can concern ourselves entirely with oriented polygons.

**Claim 10.2.1** *If we have the oriented line $\ell$ and a convex closed oriented polygon $\Delta'(\ell_1, ..., \ell_n)$, where $\ell_1, ..., \ell_n$ are oriented lines, then $|\ell|$ intersects $\Delta'$ if and only if*

- $\ell \lozenge \ell_i \geq 0$ *holds* $\forall i$ *in* $\{1, \ldots, n\}$ *or*
- $\ell \lozenge \ell_i \leq 0$ *holds* $\forall i$ *in* $\{1, \ldots, n\}$.

Furthermore, if we know the orientation of the line and the direction of winding of the polygon, we can get by with having to test only one of these cases as follows:
Consider the case we just described involving our polyhedral scene. If we know that an oriented line stabs into one of our polyhedra through one of its faces, then we know that it has to stab out of the polyhedron through another of its faces, and since we partitioned all of the space into convex regions, ignoring the case where our region is unbounded, it must then stab into one of the neighboring polyhedra via one of its portals or intersect a polygon from our original scene that is a face of the current polyhedron. And so, we need only test against the incoming portals of each of the neighboring nodes excluding the one we stabbed in from. We know these portals will be oriented counterclockwise. If none of these are intersected, then the ray intersects a polygon attached to the current node. Appealing to the right hand rule, this means that if the portal of the neighboring node has

edges $\ell_1, ..., \ell_n$ in counterclockwise order, then we only have to verify that $\ell \Diamond \ell_i \geq 0$ holds for $i \in \{1, ..., n\}$ and can ignore the other case.

The analysis in this section has been primarily directed at establishing results for the following section. Far more efficient algorithms exist for solving these queries. See Pellegrini[12] for more information.

# Chapter 11: Stabbing Lines

## 11.1 Stabbing sets of lines

**Definition 11.1.1** *If a line $\ell$ intersects a set of points $X$, then we say $\ell$* ***stabs*** *$X$.*

**Definition 11.1.2** *Given a set $T$ of sets of points, where $\ell$ intersects every set in $T$, we say that the line $\ell$ is a* ***stabbing line*** *for the set $T$.*

Clearly the concept of a stabbing line and the concept of transversal are related. However, while every transversal of a set of lines is a stabbing line for the same set, the converse is not true.

The first class of problems that we will consider involves questions pertaining to the number of lines that exist that stab a particular set of lines in space. One problem in this class was tackled by Teller and Hohmeyer[21] (determining the lines stabbing through four lines in space), and the solution is built upon in Teller's dissertation[19] to very useful effect. Some of these results can be obtained by purely geometric means as well. However, they serve to develop additional properties we exploit when working with Plücker and Study coordinates.

**Lemma 11.1.3** *Let $\ell$ and $\ell'$ be distinct lines with Plücker line coordinates $(\xi_{ij})$ and $(\xi'_{ij})$ respectively. If $\ell \Diamond \ell' = 0$ then $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ are not contained in a quadratic subspace of $(Q, F^6)$ with signature $(1,3)$ or $(3,1)$.*

*Proof:* Assume $\ell$ and $\ell'$ are distinct lines with Plücker line coordinates $(\xi_{ij})$ and $(\xi'_{ij})$ respectively such that $\ell \Diamond \ell' = 0$ and that $\boldsymbol{\xi}$ and $\boldsymbol{\xi}'$ lie in a quadratic subspace of signature $(1,3)$. We know that $B(\boldsymbol{\xi}', \boldsymbol{\xi}) = 0$, and because $\ell$ and $\ell'$ are lines, $Q(\boldsymbol{\xi}) = Q(\boldsymbol{\xi}') = 0$. Applying Lemma 5.3.2 we see that $\boldsymbol{\xi} = C\boldsymbol{\xi}'$. This implies that $\ell = \ell'$, which contradicts the assumption that $\ell$ and $\ell'$ are distinct lines. Similarly we can employ Corollary 5.3.3 to handle the case where the signature is $(3,1)$. ■

Here we adopt the convention that $\mathbf{v}^n$ refers to the $n$th vector in an ordered set, as opposed to raising to $n$th power, so that we may refer to the components of our vectors or to particular matrices and have the meanings of the subscripts be unambiguous.

**Lemma 11.1.4** *Given a set $\mathcal{L} = \{\ell_1, ..., \ell_4\}$ of lines in* $\mathbf{PG}(3, \mathbb{R})$*, if two lines in $\mathcal{L}$ intersect, then there exists a stabbing line for $\mathcal{L}$.*

*Proof:* Without loss of generality, assume $\ell_1$ and $\ell_2$ intersect at $P$. Then $\ell_3$ and $\ell_4$ must either be skew or not skew. If they are skew, then there exists a line intersecting $\ell_3$, $\ell_4$, and $P$. Otherwise $\ell_3$ and $\ell_4$ lie in a plane, and hence there exists a point $Q$ incident with both $\ell_3$ and $\ell_4$. Then $P \vee Q$ intersects all four lines. ■

**Definition 11.1.5** *For notational convenience in the following proofs we define*

$$B'(\mathbf{a}, \mathbf{b}) := \mathbf{a}^T [B]_\gamma \mathbf{b}. \tag{11.1.1}$$

**Lemma 11.1.6** *Given four lines in* $\mathbf{PG}(3, F)$ *with linearly independent Plücker line coordinates, there are either zero, one, or two lines that intersect all four. If there are two lines, then these lines do not intersect each other.*

*Proof:* Let $(\xi^k_{ij})$, where $k \in \{1, ..., 4\}$, be Plücker line coordinates for four lines in $PG(3, F)$ where the four vectors $\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^4$ are linearly independent. Let $\ell$ be a line with Plücker line coordinates $(\xi_{ij})$ that intersects all of them. Then we obtain for $k \in \{1, ...4\}$ that

$$(\boldsymbol{\xi}^k)^T B \boldsymbol{\xi} = 0. \tag{11.1.2}$$

We also know that

$$\boldsymbol{\xi}^T B \boldsymbol{\xi} = 0 \tag{11.1.3}$$

must hold for $\ell$ to be a line. Changing basis to view this problem in Study coordinates, we see that if we let $(\mathbf{u}^k, \mathbf{v}^k) = [\boldsymbol{\xi}^k]_\gamma$, then the following constraints must hold:

$$A[\boldsymbol{\xi}]_\gamma := \begin{bmatrix} u_1^1 & u_2^1 & u_3^1 & -v_1^1 & -v_2^1 & -v_3^1 \\ u_1^2 & u_2^2 & u_3^2 & -v_1^2 & -v_2^2 & -v_3^2 \\ u_1^3 & u_2^3 & u_3^3 & -v_1^3 & -v_2^3 & -v_3^3 \\ u_1^4 & u_2^4 & u_3^4 & -v_1^4 & -v_2^4 & -v_3^4 \end{bmatrix} [\boldsymbol{\xi}]_\gamma = 0. \tag{11.1.4}$$

Since the Plücker line coordinates (and hence the Study coordinates) of our lines are not linearly dependent, $\dim(\ker(A)) = 2$. Choose an orthonormal ordered basis $(\mathbf{n}^1, \mathbf{n}^2)$ for $\ker(A)$ that diagonalizes the quadratic form as in Definition 5.2.1. Any solution $[\boldsymbol{\xi}]_\gamma$ must be a linear combination of the elements of this basis, so $[\boldsymbol{\xi}]_\gamma = a\mathbf{n}^1 + b\mathbf{n}^2$. Inspecting the possibilities for the matrix $C = (B(\mathbf{n}^i, \mathbf{n}^j))_{2\times 2}$ we have one of the following cases:

$$C = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{11.1.5}$$

The first case is anisotropic and admits no nonzero solutions. In the second case

$\boldsymbol{\xi} \in \text{span}\{\mathbf{n}^1\}$. In the last case if $[\boldsymbol{\xi}]_\gamma = \mathbf{n}^1 \pm \mathbf{n}^2$ then

$$\begin{aligned} B'([\boldsymbol{\xi}]_\gamma, [\boldsymbol{\xi}]_\gamma) &= B'(\mathbf{n}^1 \pm \mathbf{n}^2, \mathbf{n}^1 \pm \mathbf{n}^2) \\ &= B'(\mathbf{n}^1, \mathbf{n}^2) \pm 2B'(\mathbf{n}^1, \mathbf{n}^2) + B'(\mathbf{n}^2, \mathbf{n}^2) \\ &= 1 - 1 = 0 \end{aligned} \tag{11.1.6}$$

so in this case there are two lines that intersect our four given lines. To see that these lines do not intersect each other, consider

$$B'(\mathbf{n}^1 + \mathbf{n}^2, \mathbf{n}^1 - \mathbf{n}^2) = B(\mathbf{n}^1, \mathbf{n}^1) - B(\mathbf{n}^2, \mathbf{n}^2) = 1 - (-1) = 2 \neq 0. \tag{11.1.7}$$

So there are zero, one, or two lines that intersect four lines with linearly independent Plücker ray coordinates, and these lines do not intersect. ∎

**Lemma 11.1.7** *If the line $\ell$ stabs the set $\mathcal{L} = \{\ell_1, ..., \ell_n\}$ of lines in* $\mathbf{PG}(3, F)$ *then $\ell$ intersects any line with Study coordinates that are a linear combination of the Study coordinates for lines in $\mathcal{L}$.*

*Proof:* Let $\ell$ be a line in $\mathbf{PG}(3, F)$ with Study coordinates $[\boldsymbol{\xi}]_\gamma$. Let $[\boldsymbol{\xi}_1]_\gamma, ...[\boldsymbol{\xi}_n]_\gamma$ be Study coordinates for lines in $\mathcal{L}$. Suppose $\ell$ stabs $\mathcal{L}$. Then $[\boldsymbol{\xi}]_\gamma[B]_\gamma[\boldsymbol{\xi}_i] = 0$ holds for $i \in \{1, ..., n\}$. Linearity yields that $[\boldsymbol{\xi}]_\gamma[B]_\gamma[\boldsymbol{\xi}']_\gamma = 0$ for any $\boldsymbol{\xi}'$ that is a linear combination of $\boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_n$. Hence $\ell$ intersects any line with Study coordinates that are a linear combination of the Study coordinates for lines in $\mathcal{L}$. ∎

**Corollary 11.1.8** *If the line $\ell$ stabs the set $\mathcal{L} = \{\ell_1, ..., \ell_n\}$ of lines in* $\mathbf{PG}(3, F)$ *then $\ell$ intersects any line with Plücker ray (or axial) coordinates that are a linear combination of the Plücker ray (or axial) coordinates for lines in $\mathcal{L}$.*

This allows us to discard the individual lines in the set $\mathcal{L}$ of lines in $\mathbf{PG}(3, F)$ and simply test for intersection against any arbitrary basis for the span of their Study coordinates.

**Corollary 11.1.9** *Let $\mathcal{L}$ be a set of lines in $\mathbf{PG}(3, F)$. Let $\beta = \{\boldsymbol{\beta}_0, ..., \boldsymbol{\beta}_n\}$ be a basis for the n-flat*

$$\bigvee_{\ell \in \mathcal{L}} p(\ell) \tag{11.1.8}$$

*in $\mathbf{PG}(5, F)$. If a line $\ell$ with Study coordinates $[\boldsymbol{\xi}]_\gamma$ satisfies*

$$[\boldsymbol{\beta}_i]_\gamma^T [B]_\gamma [\boldsymbol{\xi}]_\gamma = 0 \tag{11.1.9}$$

*for $i \in \{0, ..., n\}$, then $\ell$ stabs $\mathcal{L}$.*

Note that the basis vectors in the above corollary need not actually satisfy the Plücker quadratic form themselves, and so may not represent lines in space.

**Theorem 11.1.10** *Let $\mathcal{L} = \{\ell_1, ..., \ell_4\}$ be a set of lines in $\mathbf{PG}(3, \mathbb{R})$ with Plücker ray coordinates $(\boldsymbol{\xi}_{ij}^k)$ and consider the $4 \times 4$ matrix $(B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j))$.*

- *If $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)| < 0$, then there is no stabbing line for $\mathcal{L}$.*
- *If $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)| = 0$ and $\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^4$ are linearly independent then there is exactly one stabbing line for $\mathcal{L}$.*
- *If $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)| > 0$, then there are exactly two distinct stabbing lines for $\mathcal{L}$.*
- *If $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)| = 0$ and $\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^4$ are linearly dependent then there exists an infinite number of distinct stabbing lines for $\mathcal{L}$.*

*Proof:* Let $\mathcal{S} = (Q, \operatorname{span}\{\boldsymbol{\xi}^1, ..., \boldsymbol{\xi}^4\})$ be a quadratic space where $Q$ is the Plücker quadratic form. Let $\mathbf{d}$ represent the entries on the diagonal of the diagonalized form of $(B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j))$.

The trace of $(B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j))$ is the same as $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)|$ up to a positive scalar multiple, so we need only consider the product $\delta$ of the factors in $\mathbf{d}$.

Let $\mathcal{S}^\Delta$ denote the quotient space of vectors from $(Q, \mathbb{R}^6)$ orthogonal to $\mathcal{S}$ under $Q$.

If $\delta < 0$ then $\mathbf{d} = (1, -1, -1, -1)$ or $\mathbf{d} = (1, 1, 1, -1)$. Hence $\mathcal{S}$ has signature $(1, 3)$ or $(3, 1)$. Because $(Q, F^6)$ has signature $(3, 3)$, it follows that the quotient space has signature $(2, 0)$ or $(0, 2)$. In either case, the quotient space is anisotropic, so there can exist no lines with Plücker ray coordinates contained in it. So no lines can exist that stab $\mathcal{L}$.

If $\delta > 0$ then $\mathbf{d} = (1, 1, -1, -1)$, so $\mathcal{S}$ has signature $(2, 2)$ and the quotient space $S^\Delta$ must have signature $(1, 1)$. Let $\beta = \{\boldsymbol{\beta}_0, \boldsymbol{\beta}_1\}$ be a basis for the quotient space. Then $Q(\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1) = Q(\boldsymbol{\beta}_0 - \boldsymbol{\beta}_1) = 0$, so $\boldsymbol{\beta}_0 + \boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_0 - \boldsymbol{\beta}_1$ can each be used to specify matrices of Plücker ray coordinates for the two distinct stabbing lines of $\mathcal{L}$. Excluding scalar multiples, any other vector $\mathbf{v}$ either fails to satisify $Q(\mathbf{v}) = 0$, and so does not represent a line, or does not lie in the quotient space and so its associated line does not stab $\mathcal{L}$.

If $\delta = 0$ and $\dim(\mathcal{S}) = 4$ then $\mathbf{d}$ contains a zero, in which case the quotient space has signature $(0, 1)$ or $(1, 0)$. Then the basis vector for the quotient space that maps to zero under $Q$ can be used to specify a matrix of Plücker ray coordinates for a line that stabs $\mathcal{L}$.

If $\delta = 0$ and $\dim(\mathcal{S}) < 4$ then there are at most three linearly independent lines among $\ell_1, ..., \ell_4$. For fewer than three linearly independent lines, finding a stabbing line is trivial, so we move on to consider the case where three are linearly independent. Without loss of generality assume that $\ell_1, ..., \ell_3$ are linearly independent. Then either $\ell_1, ..., \ell_3$ are skew or they are not skew. If they are not skew, then two of them lie in a plane, and hence share a common point, so a line through that point and any point on the remaining line stabs $\mathcal{L}$. Otherwise, every point on $\ell_1$ yields a unique transversal of the regulus formed by $\ell_1, ..., \ell_3$. In either case there are infinitely many lines that stab every line in the set. ■

**Lemma 11.1.11** *Given four lines $\ell_1, ..., \ell_4$ in $\mathbf{PG}(3,\mathbb{R})$ with Plücker ray coordinates $(\boldsymbol{\xi}^k_{ij})$ respectively, if two lines from $\{\ell_1, ..., \ell_4\}$ intersect then $|B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j)| \geq 0$.*

*Proof:* If two lines in $\ell_1, ..., \ell_4$ intersect then they cannot lie in a quadratic space of signature $(1,3)$ or $(3,1)$. Therefore the diagonal of the diagonalized form of $(B(\boldsymbol{\xi}^i, \boldsymbol{\xi}^j))$ cannot be $(1,-1,-1,-1)$ or $(1,1,1,-1)$, which are the only cases in which the determinant is negative, so the diagonal must either contain a zero, in which case the determinant is zero, or it must equal $(1,1,-1,-1)$, which is greater than 0. ■

**Theorem 11.1.12** *The number of lines that intersect every member of a set of lines in $\mathbf{PG}(3,\mathbb{R})$ is either zero, one, two, or uncountably infinite.*

*Proof:* Let $\mathcal{L}$ be a set of lines in $\mathbf{PG}(3,\mathbb{R})$. Let $(\xi^k_{ij})$ be Plücker ray coordinates for the lines in $\mathcal{L}$. Let $U$ be the quotient space of $(Q,\mathbb{R}^6)$ with respect to $\{\boldsymbol{\xi}^k\}^n_{k=1}$. Since $U$ is a quadratic subspace, we appeal to Theorem 5.4.1 to determine that there are zero, one, two or an uncountably infinite number of one-dimensional linear subspaces, such that $\mathbf{u} \in U$ implies $Q(\mathbf{u}) = 0$. Given any such subspace, any nonzero vector contained in it determines the Plücker ray coordinates for the same line, and each subspace maps onto a distinct line, so there are zero, one, two or infinitely many lines that intersect every member of a finite set of lines. ■

These results enable us to consider the second class of stabbing line problem, which involves finding lines that stab through sets of polygons.

## 11.2 The extremal stabbing line theorem

**Definition 11.2.1** *We define an **extremal stabbing line** for a set of polyhedra to be a line that intersects every polyhedron in the set and intersects four of the edges of the polyhedra in the set.*

This is somewhat contrary to a similar definition given by Pellegrini,[9] which also includes the stipulation that the stabbing line must be tangent to every polyhedron in the set.

The following theorem motivates the consideration of extremal stabbing lines.

**Theorem 11.2.2** ***(Extremal Stabbing Line Theorem)*** *Let $\mathcal{P}$ be a set of open polygons, where $\mathcal{P}$ contains more than one polygon. Let $\mathcal{P}'$ be the set of the closures of polygons in $\mathcal{P}$. If there exists a line $\ell$ that intersects every polygon in $\mathcal{P}$, then there exists a line $\ell'$ that intersects every polygon in $\mathcal{P}'$ and intersects four of the lines that border those polygons, of which at most two lines come from any given polygon in $\mathcal{P}$.*

*Proof:* Assume there exists a stabbing line $\ell$ for a set $\mathcal{P}$ of open polygons and that $\mathcal{P}$ contains more than one polygon. Then $\ell$ lies in a cell $C$ of the arrangement $A(\mathcal{H}_{\mathcal{P}})$.

We will make heavy use of the following construction in order to justify the fact that we can pivot our line around a point while moving the other point in a constant direction within one of the polygons of $\mathcal{P}$ (or one of the edges of those polygons) until we intersect an edge of one of our polygons, either by having moved to the boundary of the polygon in which we are moving, or by the act of pivoting causing our constructed line to intersect a line on the boundary of another polygon somewhere along the way.

Let $L_{XY} : x \mapsto L_{XY}(x)$ be a continuous one-to-one parametrization of the line segment between $X$ and $Y$ such that $L_{XY}(0) = X$ and $L_{XY}(1) = Y$. Let

$$M(X, Y, Z, C') := \{x \in [0, 1] : p(L_{XY}(y) \vee Z) \in C', \forall y \leq x\}, \qquad (11.2.1)$$

where $X$, $Y$, and $Z$ are points, Z not contained in the line $X \vee Y$, $Y$ is a point such that $\pi(Y)$ is one of the hyperplanes in the arrangement that containing $C$, and $C'$ is a closed cell containing $p(L_{XY}(0) \vee Z)$. Then because $0 \in M(X, Y, Z, C')$ implies $M(X, Y, Z, C')$ is not empty and by construction it is bounded above by 1, we know $\sup M(X, Y, Z, C')$ exists.

Moreover by the closure of the sets and continuity of the operations involved, $\sup M(X, Y, Z, C') \in M(X, Y, Z, C')$. This tells us $p(L_{XY}(\sup(M, X, Y, Z, C')) \vee Z) \in \partial C$, so this point lies on a hyperplane bordering $C$. Let $\ell(X, Y, Z, C')$ denote the line in $\mathbf{PG}(3, \mathbb{R})$ represented by this point and let $L(X, Y, Z, C')$ denote the line represented by this hyperplane.

Let $\Delta_m$ and $\Delta_n$ be distinct open polygons in $\mathcal{P}$, let $P$ be a point on $\ell$ in $\Delta_m$, and let $Q$ be a point on $\ell$ in $\Delta_n$. Choose a point $P_0$ in $\partial\Delta_m$. Let $\ell_1 := \ell(P, P_0, Q, C')$ and $\ell_a := L(P, P_0, Q, C')$. Then $\ell_1 \Diamond \ell_a = 0$ and $\ell_1 \in C'$.

Let $\Delta_a$ be a polygon in $\mathcal{P}$ bounded in part by $\ell_a$, let $P_1$ be a vertex of $\Delta_a$ contained in $\ell_a$, let $P_a = \ell_1 \wedge \ell_a$, and let $\mathcal{H}_1$ denote the set of four-dimensional hyperplanes obtained by intersecting the hyperplanes of $\mathcal{H}_{\mathcal{P}} \backslash \{\pi(\ell_a)\}$ with the hyperplane $\pi(\ell_a)$. Let $C_1' = C' \cap \pi(\ell_a)$. Then $C_1'$ is an intersection of closed sets and so is closed. Furthermore $C_1'$ is a closed cell in the arrangement $A(\mathcal{H}_1)$. Let $\Delta_1$ be polygon distinct from $\Delta_a$ and $Q_1$ be a point common to $\ell_1$ and $\Delta_1$. Let $\ell_2 := \ell(P_a, P_1, Q_1, C_1')$ and $\ell_b := L(P_a, P_1, Q_1, C_1')$. Then $\ell_2 \Diamond \ell_a = \ell_2 \Diamond \ell_b = 0$ and $\ell_2 \in C'$.

If $P_1 \in \ell_2$, then $\ell_2$ intersects two edges of the same triangle. Otherwise, it intersects one edge on two distinct triangles. In the former case, we fix $P_1$ to avoid losing contact with the two edges we have contacted so far as we did with $Q$ before, and repeat this process to move a point on a different triangle towards its edges to make contact, yielding a line incident with three edges, and then move the line within the edge we reach to contact a fourth edge. In the latter case, we fix one point and pivot within the other line segment until we intersect another line, then repeat with the other.

In either case, we obtain a line $\ell'$ that intersects each of four lines $\ell_a, ..., \ell_d$ that border the polygons of $\mathcal{P}$, and have $\ell' \in C'$, so $\ell'$ intersects the closure of each polygon in $\mathcal{P}$.

The only case that requires further explanation is when this procedure results in the intersection of our constructed line with three lines from distinct triangles. In this case, either the three lines are all mutually skew or they are not all skew.

If they are not skew, then two of them line in a plane, and the other must intersect this plane at some point *P*. Moreover, in order for our line to stab through all three it must line in this plane and already pass through the point *P* and both lines. So in this case we pivot around the point *P*, while heading towards one of the vertices of a triangle bounded in part by one of the other two lines.

Otherwise, the three lines are all skew, so we have a regulus. We construct our parameterization as before, moving along one of the lines in the regulus towards a shared point, but instead of pivoting around a point as we have done so far, we map this point to the transversal of the regulus passing through our parameterized point and appeal to the continuity of this mapping.

See Teller and Hohmeyer[20] for an alternate proof. ■

More succinctly, given a line that stabs through interior of each polygon in a set of closed polygons, we can adjust the line until it intersects four of the edges of those polygons, in such a way that it will still intersect those closed polygons. As a result of this process, the intersections with the newly constructed line may occur on their boundary.

### 11.3 Stabbing sets of oriented polygons

Recall from the discussion of ray-casting in the previous chapter that we can employ the structure of a scene that is decomposed into cells to have to consider only one of the two possibilities when stabbing polygons in a scene. This is one of many contexts that gives rise to the problem of trying to find out if there exists a line that can stab through a set of oriented polygons.

By stabbing a set of oriented polygons, we mean that we know that if the line stabs through the first polygon by satisfying $\ell \lozenge \ell_i \geq 0$ for each edge $\ell_i$ of the polygon, then it will stab through every other polygon in the set in the same manner. In other words, no intersection test with an oriented polygon with edges $\ell'_i$ from the set will require $\ell \lozenge \ell'_i \leq 0$ to hold for all edges.

Determining whether or not there exists a line that stabs through a set of oriented polygons at first appears to be an exercise in root finding. We are interested in whether or not there exists a root $\boldsymbol{\xi}$ to the Plücker quadric in a region defined by a number of linear constraints, given in the form of the Plücker ray coordinates $\boldsymbol{\xi}^k$ of the edges of our polygons. In other words, we are searching for a solution to

$$\boldsymbol{\xi}^T [B]_\beta \boldsymbol{\xi} = 0 \tag{11.3.1}$$

subject to the constraints

$$(\boldsymbol{\xi}^k)^T [B]_\beta \boldsymbol{\xi} \geq 0, \tag{11.3.2}$$

for $k \in \{1, ..., n\}$.

This is similar to a quadratic programming problem. We know that the diagonal form of $B$ has signature $(3, 3)$. Our underlying quadratic form is neither positive semi-definite nor negative semi-definite. This means that traditional convex quadratic programming techniques are not useful. There exists a technique called Ritter's cutting plane method[2, 13] for solving non-convex quadratic programming problems that can be modified to solve this problem. However, Ritter's method works poorly in spaces with multiple positive and negative eigenvalues.

We appeal to the contrapositive of the Extremal Stabbing Line Theorem to transform some of our inequalities into equalities. If there does not exist an extremal stabbing line for our set of polygons, then there does not exist a stabbing line for the polygons.

We don't know necessarily which edges will be equalities, but even naively there are at most $\binom{n}{4}$ possibilities for $n$ edges. We do know that at most two of them can come from the same polygon, and if we will be performing many of these checks on related polygon sets, we can use dynamic programming techniques to retain the constructed basis for the quotient space of a given set of equalities.

So with $n$ polygons, that have at most $k$ edges each, we consider

$$\binom{n}{4}k^4 + \binom{n}{2}k^2 + \binom{n}{1}\binom{n-1}{2}k^2 \tag{11.3.3}$$

cases or, more concisely, $O(n^4)$ cases if we approach the problem in the most straightforward manner. This happens since we can have either four edges from distinct polygons, two consecutive edges each from two polygons, or two consecutive edges from one and one edge each from two others. In literature on the subject, these cases are typically labeled $EEEE$, $VV$, and $VEE$ respectively, where $V$ and $E$ indicate whether each intersection occurs along an edge or at a vertex. Note that when we are selecting two edges from the same polygon, we have to choose only one, because they must be consecutive.

Now, ignoring degenerate cases, the problem simply becomes a matter of choosing four of our edges to consider as equalities and finding an orthogonal basis for our quotient space so that Equation 11.1.6 holds and testing whether or not one of our two candidate solutions satisfies the remaining inequalities associated with our other polygon edges.

If one of the candidate solutions satisfies all of our inequalities, then there exists a line that stabs through the supplied set of oriented polygons and you know one of the extremal stabbing lines for the polygon set.

If none of the candidate solutions pass the system of inequalities, then there does not exist a line that stabs through the polygon set, by the contrapositive of the extremal stabbing line theorem.

For simplicity, we have ignored the cases where numerical degeneracies and edges lying in the same ruled surface cause the proposed algorithm to fail when implemented in software. Usually we are concerned with obtaining a conservative approximation of visibility in the scene. When unable to calculate a 2-dimensional orthogonal basis for the quotient space due to linear dependence between $\boldsymbol{\xi}^1, \ldots, \boldsymbol{\xi}^4$, one can appeal to "jittering" to enlarge one or more of our polygons slightly to break linear dependence. Similarly, it is not practical to calculate exactly in computer graphics, so similar compromises are typically made by normalizing our Plücker coordinates and allowing a small $\epsilon$ region around zero to count as equality in intersection testing.

### 11.4 Applications of quadratic programming

Ritter's cutting plane[2, 13] finds global minima of a quadratic form subject to linear constraints. It accomplishes this by finding local minima and restructuring the problem into one involving a new quadratic form and constraints that do not contain each local minima in turn. It includes special handling for the cases where the values obtained by the quadratic form are unbounded in the constrained region.

Its primary weakness is that it can be very slow when working in quadratic spaces of signature (n,m) where $n \geq 2, m \geq 2$ gracefully.

However, we are concerned solely with knowing of the existence of a root. It suffices for our purposes to find a single vector that evaluates to 0 under the quadratic form. If we find any pair of vectors in our quadratic subspace of which one yields a positive value and the other a negative value under our quadratic form, we can appeal to the intermediate value theorem to know that there must exist a vector that can be obtained by interpolating between them that satisfies our constraints. Since the region is convex this vector must satisfy our constraints.

To overcome the problem with signature, we can fix two of our constraints as equalities in the same manner as we fixed four of them in the previous section. The resulting quadratic subspace will be of signature $(1, 3)$, $(3, 1)$ or $(2, 2)$. In the former two cases we may now freely employ Ritter's method. In the last case we must fix a third constraint in order to employ Ritter's method, and because we would have to take at least as many operations as it takes to evaluate the fourth constraint to use Ritter's method, we simply mark this case for later processing and try the next selection. If we exhaust all of our possible constraints, then we come back to the ones we marked and apply the procedure described in the previous section.

Because Ritter's method finds numerous local minima on the way to finding the global minimum, we can stop as soon as we find a local minimum that evaluates to less than zero. If it tells us that the quadratic form is unbounded below on constrained region, we know there exists a point inside of it that evaluates to a negative value. If it yields a global minimum greater than zero, then this combination of fixed constraints has no solutions. Then we apply our modified form of Ritter's method to obtain a local maximum that evaluates to greater than zero in the same region or to determine if a point exists that evaluates to a positive value. Similar to before, if Ritter's method yields the global maximum less than zero, then this combination of fixed constraints has no solutions. With both of these points in hand (or at least knowledge of their existence) we can appeal to the intermediate value theorem to obtain the fact that a solution exists.

While the theoretical upper bound on the number of operations is at least as high as in the previous case, this method is more likely to yield a solution early in the processing. Furthermore, as we are only forcing the solution to coincide with two of the edges, we have fewer numerical stability problems.

### 11.5 Accelerating the search for stabbing lines

An alternative hybrid approach is to take the Las Vegas approach of checking to see if the first local maximum and minimum obtained by an arbitrary quadratic programming algorithm such as Beale's algorithm[2] are on different sides of zero and appeal to the intermediate value theorem to obtain an early answer before we turn to the approach from Section 11.3. In the event of a hit, this process can gain a considerable performance benefit for a low amount of additional overhead.

However, unlike the first method given, these quadratic programming methods cannot be used to exhaustively enumerate the extremal stabbing lines for a set of polygons, a process that is important for constructing structures such as Durand's visibility skeleton.[5]

# Appendix A: Assorted Theorems and Observations

Finally, we would like to point to a number of miscellaneous theorems and conjectures that tie in nicely with the other work here on Study coordinates, but that are of only cursory interest or would have required the inclusion of too much supplementary material to be effectively presented here.

## A.1 Plücker and Study coordinates for polar lines

**Lemma A.1.1** *Let $\ell$ be an (oriented) line in* $\mathbf{OPG}(3, F)$ *with Plücker ray coordinates* $(\xi_{ij})$*. Then the Plücker axial coordinates* $(\Xi_{ij})$ *for its orthogonal complement* $\ell^{\Delta}$ *are the same as* $(\xi_{ij})$ *up to a nonzero (positive) scalar multiple.*

*Proof:* Let $\ell = P_{\mathbf{v}} \vee P_{\mathbf{w}}$ be an (oriented) line in $\mathbf{OPG}(3, F)$ with Plücker ray coordinates $(\xi_{ij})$. Let $(\Xi_{ij})$ be the Plücker axial coordinates of $\ell^{\Delta}$. Because a correlation preserves incidence $\ell^{\Delta} = P_{\mathbf{v}}^{\vdash} \wedge P_{\mathbf{w}}^{\vdash} = \Pi_{\mathbf{v}} \wedge \Pi_{\mathbf{w}}$. The Plücker axial coordinates of $\Pi_{\mathbf{v}} \wedge \Pi_{\mathbf{w}}$ are then specified in terms of the minor determinants of the $2 \times 2$ matrix with rows $\mathbf{v}$ and $\mathbf{w}$. A matrix of Plücker ray coordinates for $\ell$ are defined by the same determinants, so $(\xi_{ij}) = C(\Xi_{ij})$ for a nonzero (positive) scalar $C$. ∎

**Corollary A.1.2** *If* $(\xi_{ij})$ *are Plücker ray coordinates for a line* $\ell$ *in* $\mathbf{OPG}(3, F)$ *and*

$$\boldsymbol{\xi} = (\mathbf{u}, \mathbf{v}), \tag{A.1.1}$$

*where* $\mathbf{u}, \mathbf{v} \in F^3$ *then a matrix of Plücker ray coordinates* $(\xi'_{ij})$ *for* $\ell^\Delta$ *is determined by*

$$\boldsymbol{\xi}' = (\mathbf{v}, \mathbf{u}). \tag{A.1.2}$$

*Proof:* This is an immediate consequence of the previous lemma and Theorem 6.3.1.

■

**Theorem A.1.3** *If* $\{P_{\mathbf{u}}; P_{\mathbf{v}}\}$ *is a Study representative pair for a line* $\ell$*, then* $\{P_{\mathbf{u}}; -P_{\mathbf{v}}\}$ *is a Study representative pair for the line* $\ell^\Delta$*, which is the orthogonal complement to* $\ell$*.*

*Proof:* Recall that if $\boldsymbol{\xi} = p(\ell) = (\mathbf{a}, \mathbf{b})$ then $p(\ell^\Delta) = (\mathbf{b}, \mathbf{a})$. Expressing these in terms of the new basis, we obtain the representative pairs $\{P_{\mathbf{a+b}}; P_{\mathbf{a-b}}\}$ and $\{P_{\mathbf{a+b}}; P_{\mathbf{b-a}}\}$ respectively.

■

## A.2 Isotropic vectors in the quadratic space $(Q, F^6)$

Study coordinates are homogeneous, so a positive scalar multiple of a column of Study coordinates for an oriented line is also a column of Study coordinates for the same oriented line. This allows us to normalize our Study coordinates to obtain a more useful form. We make the somewhat odd choice to normalize them so that the magnitude of each column of Study coordinates is $\sqrt{2}$ and investigate the consequences of this choice.
We would like to pause for a moment to consider what the space of points on the Plücker quadric looks like at a high level.

**Theorem A.2.1** *The space of isotropic vectors in the quadratic space* $(Q, F^6)$ *is the Cartesian product of two spheres.*

*Proof:* Let $\ell$ be a line with Study coordinates $[\boldsymbol{\xi}]_\gamma = (u_1, u_2, u_3, v_1, v_2, v_3)$ of magnitude $\sqrt{2}$. The values of **u** and **v** have to satisfy two equations and contain six unknowns. The first equation

$$u_1^2 + u_2^2 + u_3^2 - v_1^2 - v_2^2 - v_3^2 = 0 \tag{A.2.1}$$

is derived from the quadratic form. The second comes from the normalization, yielding

$$u_1^2 + u_2^2 + u_3^2 + v_1^2 + v_2^2 + v_3^2 = 2. \tag{A.2.2}$$

Since

$$u_1^2 + u_2^2 + u_3^2 = v_1^2 + v_2^2 + v_3^2, \tag{A.2.3}$$

we really have two equations,

$$u_1^2 + u_2^2 + u_3^2 = \langle \mathbf{u}, \mathbf{u} \rangle = 1 \tag{A.2.4}$$

and

$$v_1^2 + v_2^2 + v_3^2 = \langle \mathbf{v}, \mathbf{v} \rangle = 1. \tag{A.2.5}$$

And so the space of isotropic vectors in $(Q, F^6)$, which are of magnitude $\sqrt{2}$, is the same as the set of points on the product of two spheres.

With this recognized, we can remove the requirement that the magnitude of each vector be $\sqrt{2}$, and simply interpret the vectors **u** and **v** as homogeneous coordinates for two points on a sphere. ∎

### A.3 The space of lines in $\mathbf{PG}(3, F)$

A note should be made about the number of dimensions of information present in the set of lines in $\mathbf{PG}(3, F)$. We use a vector with six components to represent an (oriented) line $\ell$, but because we identify (nonzero) scalar multiples, we effectively lose a dimension of information to homogeneity. Moreover, because $p(\ell)$ must lie on the Plücker quadric, we lose another dimension of information. So in essence there are only four dimensions of information present in an (oriented) line in a three-dimensional space, yet we encode them with six coefficients. This explains why the "screw coordinates" used in robotics engineering are able to pack two more dimensions of information into the same six coefficients.

### A.4 Assorted results from other sources

These theorems are taken from other sources and are cited without proof with references to where to find more information on these topics.

**Theorem A.4.1** ***(Coxeter[3] p.147 Theorem 7.64)*** *If two non-parallel lines are represented by Study representative pairs $\{A; B\}$ and $\{C; D\}$ then the two common perpendiculars are represented together by $\{E; F\}$ and $\{E; -F\}$ where $E$ and $F$ are the intersections of the lines $(A \vee C)^{\Delta}$ and $(B \vee D)^{\Delta}$ with $H^{\infty}$.*

**Theorem A.4.2** ***(Coxeter p.152 Theorem 7.84)*** *Two lines are parallel without necessarily intersecting if and only if one intersects the polar of the other.*

**Remark A.4.3** *This ties into the other theorems here by giving us a test for perpendicularity between two lines with Study coordinates $[\boldsymbol{\xi}']_{\gamma}$ and $[\boldsymbol{\xi}]_{\gamma}$. These two lines* *are perpendicular (possibly without intersecting) if*

$$[\boldsymbol{\xi}']_{\gamma}[\boldsymbol{\xi}]_{\gamma} = 0. \tag{A.4.1}$$

**Theorem A.4.4** ***(Coxeter p.152 Theorem 7.85)*** *Arbitrary rotations and translations of oriented lines in* $\mathbf{OPG}(3, F)$ *can be represented in terms of quaternion rotations as a rotation applied to each of the line's Study coordinates* $\{P_{\mathbf{u}}; P_{\mathbf{v}}\}$ *by a pair of unit quaternions* $s$ *and* $t$*, where* $s$ *represents rotation, and* $t$ *translation. We obtain the Study coordinates* $\{P_{\mathbf{u}'}; P_{\mathbf{v}'}\}$ *for the transformed line as follows:*

$$\mathbf{u}' = s\mathbf{u}s^{-1}, \quad \mathbf{v}' = t^{-1}\mathbf{u}t. \tag{A.4.2}$$

More recently an article by Weiner and Wilkins[22] reviewed a similar mechanism to transform vectors in $\mathbb{E}^4$, which by extension performs a similar operation on points and planes in $\mathbf{OPG}(3, F)$ as well. This is primarily of interest because it indicates that we can translate and rotate lines without having to return to a point representation.

# Appendix B: History

## B.1 Past

Julius Plücker (1801-1868) was a German mathematician and physicist who made many contributions to synthetic and analytic geometry. His geometric studies occurred during the earliest and latest stages of his academic career, with his early work focused upon supplying geometry a more robust analytic background, and his later work focused upon what is now known as line geometry.

Felix Klein (1849-1925) received his doctorate, supervised by Plücker, in 1868. He then proceeded to complete the second part of Plücker's *Neue Geometrie des Raumes*, which had been left unfinished by Plücker's death.

One product of their labors of interest in the area of visibility is known by a variety of names, such as the Plücker quadric, or the Klein quadric[1], and provides a six-dimensional homogeneous representation for lines in three-dimensional projective geometry. This representation is particularly useful as it combines naturally with homogeneous equations for points and planes. More accurately, it is usually stated that the Plücker (or Klein) quadric is an example of a Klein quadratic set, a term used to refer to any quadratic set in five-dimensional projective space.

The Plücker quadric has been reinvented and renamed many times over the years and goes by many other names in literature, including Grassmann-Plücker coordinates, the

Grassmannian, and the Grassmann manifold $\mathcal{F}_4^2$ after Hermann Grassmann (1809-1877), the inventor of what is now known as exterior algebra.[14,15,17] These distinctions are not without merit; for instance, considering the Plücker quadric as the manifold $\mathcal{F}_4^2$ allows one to bring to bear the tools of differential geometry.

Eduard Study (1862-1930) derived what are now known as Study coordinates[3] in the progress of his research into straight lines in elliptic geometry. The Study representative pair and coordinates described herein are an oriented variation of his original work and arise naturally as the result of diagonalizing the quadratic form underlying the Plücker quadric.

## B.2 Present

There is a fairly active community using Plücker coordinates and their various mathematical cousins, such as the screw coordinates that arise in robotics. Far too much is going on or has gone before in the field to summarize here, so this list by necessity describes the work of only a small sample.

Jorge Stolfi's work on oriented projective geometry has given computational geometry a powerful set of tools for working with lines and oriented flats of arbitrary dimension. His techniques demonstrate a good balance between the concerns of abstract mathematics and practical programming. His work provides a theoretical basis for much of what is developed here.

Marco Pellegrini is generally considered the expert in the computational complexity of stabbing line algorithms and algorithms involving line complexes in general. He has a wide array of papers covering almost all aspects of this field.[9–12]

Seth Teller and Michael Hohmeyer published a number of papers using the Plücker coordinates of lines to stab oriented polygons and calculate visibility.[20,21] Their approach to the stabbing line problem uses SVD decomposition to break apart the Plücker quadratic form. Teller's Ph.D. dissertation[19] provoked a great deal of interest in the Plücker quadric in the author's social circle in particular.

Frédo Durand's work uses a powerful data structure dubbed the visibility skeleton.[5] This structure is computed using Plücker coordinates and, among other many other things, provides a way to parameterize the "shaft" of extremal lines through a set of polygons. More recently, Florent Duguet[4] has provided a way to make the calculation of the visibility skeleton more robust in the face of large numbers of coaxial hyperplanes and his approach uses Plücker coordinates heavily.

# Appendix C: Notation

The notation and terminology used throughout this thesis is culled from a wide array of sources from the last 70 years. The notation has drifted considerably in that time, and trying to present this material as a coherent whole with the emphasis shifted to using linear algebra has resulted in some compromises.

## C.1 Sources

We made a conscious decision to avoid using the convention of simply referring to a point via a particular homogenous coordinate for it and to avoid referring to lines as rescaled additions of homogeneous points as is common practice in works with a heavy projective geometry component.[6] This choice was made to avoid confusing the reader with a mixture of cases in which we allowed and disallowed scalar multiples, to avoid confusion with the meet and join operations and to emphasize that we are really only using linear algebra underneath everything.

This has the benefit that many of the equations can be expressed in terms of conventional vector operations without the profusion of strange special case symbols for geometric operations with redundant meanings or ambiguous values such as $\{xX\}$ and $(X|Y)$, which are often used to stand for the signed distance from a point to a plane.

Some of this material could have been presented more generally in the context of Stolfi's oriented projective geometry,[16–18] but that is a rather large amount of supplementary material with which to require familiarity and limiting our underlying structure to what we construct here of oriented elliptic geometry renders these results (hopefully) more accessible. In the interest of allowing the reader to progress through this material, the definitions were chosen with compatibility in mind.

The choice of representation for positive/negative orientation of flats was selected in line with the very elegant representation for an oriented projective space designed by Stolfi,[16–18] which treats oriented flats as oriented great subspheres. Stolfi's work uses a different order for Plücker line coordinates, which does not admit as elegant an interpretation for the lines that are not at infinity and complicates the transformation we use to Study coordinates. We do raid his work for effective terminology. For example, the choice of $\alpha^{\dashv}$ and $\alpha^{\vdash}$ for complements mirrors his usage, as do the terms "vacuum" and "universe."

The terms Plücker ray coordinate and Plücker axial coordinate are taken from Graustein[7] p. 462. They are used primarily in the same context as that of Veblen's notion of sense classes for *doubly oriented lines* referenced in Coxeter[3] p. 33.

The use of $\boldsymbol{\xi}$ to represent Plücker ray coordinates mimics current practice.[11,12] The choice of $\boldsymbol{\Xi}$ for Plücker axial coordinates is chosen for symmetry in a manner similar to the relations between $p_{ij}$ and $P_{ij}$ in Coxeter. This notation was chosen over that of Coxeter to clarify meaning, since we use upper case names for points, and $P$ is both a common point name and $P_{\mathbf{v}}$ is the point with homogeneous coordinates $\mathbf{v}$, that letter was accumulating too many meanings.

The names for the mappings $p(\ell)$ and $\pi(\ell)$ are taken from Pellegrini.[11]

The choice to refer to the matrix $(\xi_{ij})$ as the Plücker coordinates rather than to the specific contents of the vector $\boldsymbol{\xi}$ of six minor determinants that fully specifies $(\xi_{ij})$ is a nod to the fact that the choice of components for $\boldsymbol{\xi}$ is arbitrary.

The terminology and notation for the Study representative pair are due to Coxeter.[3] For more information see p. 150.

The choice of $M_{\boldsymbol{\xi}}(\lambda)$ as a label of the parametrization of Plücker coordinates was adapted somewhat arbitrarily from an appendix in Duguet's thesis,[4] which was the first source we found that explicitly mentioned it.

Most of the terminology about geometries that is independent of the underlying vector space is chosen to coincide with the definitions in Beutelspacher and Rosenbaum.[1]

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